On a Problem about the Connectivity of the Enhanced Proper Power Graph of a Finite Group

A. Babai & A. Mahmoudifar

1Department of Mathematics, University of Qom, Qom, Iran
2Department of Mathematics, Tehran North Branch, Islamic Azad University, Tehran, Iran

email: a_babai@aut.ac.ir & alimahmoudifar@gmail.com

Abstract

Let $G$ be a group with identity element $e$. The proper power graph and proper enhanced power graph of $G$, denoted by $\Gamma_P^*(G)$ and $\Gamma_{EP}^*(G)$, respectively. Also, the prime graph of $G$, denoted by $\Gamma_{GK}(G)$.

In [1], the authors asked which groups do have the property that $\Gamma_P^*(G)$ is connected?

In this paper, we show that if $\Gamma_{GK}(G)$ is disconnected, then $\Gamma_P^*(G)$ and $\Gamma_{EP}^*(G)$ are disconnected. Moreover, we prove that if $G$ is a nilpotent group which is not a $p$-group, then $\Gamma_{EP}^*(G)$ is a connected graph.

2000 AMS Subject Classification: 05C25, 05C69, 20D05.

Keywords: Finite group, power graph, enhanced power graph, prime graph.

1 Introduction

Throughout this paper, we suppose that $G$ is an arbitrary group with the identity element $e$. The power graph of $G$ is denoted by $\Gamma_P(G)$ (some authors denote power graph by $P(G)$ or $G(G)$). The vertex set of $\Gamma_P(G)$ is equal to $G$ and two distinct elements $x$ and $y$ of $G$ are adjacent to each other in power graph if either $x \in \langle y \rangle$ or $y \in \langle x \rangle$. In other words, $x$ and $y$ of $G$ are adjacent in $\Gamma_P(G)$, whenever either $\langle x \rangle \langle y \rangle = \langle x \rangle$ or $\langle x \rangle \langle y \rangle = \langle y \rangle$. This graph was introduced for the first time for semigroups in [6] and then it was defined for groups in [4] and [5].

Also, the commuting graph was introduced by Brauer and his coworker in [3], for the first time. We denote this graph by $\Gamma_C(G)$ and its vertex set is $G$ and two distinct elements $x$ and
y of $G$ are adjacent if and only if $x$ and $y$ commute with each other. In other words, $x$ and $y$ of $G$ are adjacent in $\Gamma_C(G)$, whenever $\langle x \rangle \langle y \rangle$ is an abelian subgroup of $G$.

Another graph that is defined on $G$, is the enhanced power graph which is denoted by $\Gamma_{EP}(G)$ (in [1], the authors denoted this graph by $G_e(G)$). Similar to the previous graphs, the vertex set of enhanced power graph is $G$. There is an edge between vertices $x$ and $y$ in $\Gamma_{EP}(G)$, whenever $\langle x \rangle \langle y \rangle$ is an abelian subgroup of $G$. This graph was introduced in [1] for the first time, in order to find that for an arbitrary group how much its power graph is closed to its commuting graph. The interested readers can be consulted [2, 8] for more information about the enhanced power graph.

It is easy to see that for a group $G$, the power graph $\Gamma_P(G)$ is a subgraph of the enhanced power graph $\Gamma_{EP}(G)$ and also $\Gamma_{EP}(G)$ is a subgraph of the commuting graph $\Gamma_C(G)$. In [1], it is characterized all finite groups $G$ such that $\Gamma_P(G) = \Gamma_{EP}(G)$, $\Gamma_P(G) = \Gamma_C(G)$ and $\Gamma_{EP}(G) = \Gamma_C(G)$.

In graph theory, a vertex of a graph is named dominating vertex if it is adjacent to every other vertices of the vertex set. For instance, suppose that $G$ is a group, the identity element $e$ is a dominating vertex of the enhanced power graph $\Gamma_{EP}(G)$ and commuting graph $\Gamma_C(G)$. Also if $G$ is a finite group, then $e$ is a dominating vertex in the power graph of $G$ and so $\Gamma_P(G)$ is a connected graph. This fact encouraged the authors in [1], to put the following question:

**Question 1.1.** Which groups do have the property that the power graph is connected when the identity is removed?

For convenience, we denote by $G^*$ the set $G \setminus \{e\}$, where $e$ is identity element of $G$. Also, we show by $\Gamma_{EP}^*(G)$ and $\Gamma_P^*(G)$ the proper enhanced power graph and proper power graph of $G$, respectively, which obtained by removing the identity element from their vertex set. Since $e$ is a dominating vertex of $\Gamma_{EP}(G)$, so as the above question, we may ask the following question:

**Question 1.2.** For which finite group $G$ the proper enhanced power graph $\Gamma_{EP}^*(G)$ is a connected graph?

If $n$ is an integer number, then the set of all prime divisors of $n$ denoted by $\pi(n)$. If $G$ is a finite group, then $\pi(G) = \pi(|G|)$. The set of all element orders of $G$ is denoted by $\omega(G)$. it is clear that $\omega(G)$ is closed and partially ordered by the divisibility relation. So $\omega(G)$ is uniquely determined by the set $\mu(G)$ of elements which are maximal under the divisibility relation.
The prime graph (or Gruenberg-Kegel graph) of $G$ denoted by $\Gamma_{GK}(G)$ is another graph associated to a group $G$. Its vertex set is equal to $\pi(G)$ and two distinct prime numbers $p$ and $q$ of $\pi(G)$ are adjacent if and only if $pq \in \omega(G)$.

In this paper, we show that if prime graph $\Gamma_{GK}(G)$ is disconnected, then proper power graph $\Gamma_p^*(G)$ and proper enhanced power graph $\Gamma_{EP}^*(G)$ are disconnected. Since the prime graphs of all finite simple groups are well known, so we produce a method to consider the answer of Questions 1.1 and 1.2 in the class of all finite simple groups. However, first of all, for getting this result, we have to define the following graphs.

**Definition 1.3.** Let $G$ be a group and $MC(G)$ be the set of all maximal cyclic subgroups of $G$. We denote $\Gamma_{MC}(G)$ as a graph which its vertex set is $MC(G)$ and two distinct vertices $C_1$ and $C_2$ are adjacent in $\Gamma_{MC}(G)$ if $C_1 \cap C_2 \neq e$.

**Definition 1.4.** Let $G$ be a group. We denote $\Gamma_\mu(G)$ as a graph whose vertex set is $\mu(G)$ and $m, n \in \mu(G)$ are adjacent in $\Gamma_\mu(G)$ if $(m, n) \neq 1$.

In this paper, we check the connectivity of proper enhanced power graph and find some statements about the connectivity of proper enhanced power graph. Also, we will prove that if the prime graph of a finite group is disconnected, then the proper enhanced power graph of a finite group is disconnected.

Throughout this paper, $G$ is a finite group. We denote the order of an element $x$ by $o(x)$. Moreover, all further unexplained notation is standard and referred to [7].

## 2 Prime graph and enhanced power graph

In this section, we try to find a relation between the prime graph and proper enhanced power graph of a finite group. In particular, we will prove that if the prime graph of a finite group is disconnected, then the proper enhanced power graph of a finite group is disconnected too.

**Theorem 2.1.** Let $G$ be a finite group. Then $\Gamma_{EP}^*(G)$ is connected if and only if $\Gamma_{MC}(G)$ is connected.

**Proof.** First, let $\Gamma_{EP}^*(G)$ be connected. Also, assume that $C$ and $T$ are two distinct maximal cyclic subgroups of $G$. Let $C = \langle x \rangle$ and $T = \langle y \rangle$, for some $x, y \in G^*$ and $x \neq y$. Since $\Gamma_{EP}^*(G)$ is connected, so there exists a path between $x$ and $y$ in $\Gamma_{EP}^*(G)$. We consider

$$x = u_1 \sim u_2 \sim \ldots \sim u_n = y,$$

3
as a path between $x$ and $y$ in $\Gamma^{\ast}_{EP}(G)$. Suppose that $T_i$ is a maximal cyclic subgroup of $G$ which
is contain $u_i$, for every $1 \leq i \leq n$. Since $u_j \sim u_{j+1}$ in $\Gamma^{\ast}_{EP}(G)$, so there exists a maximal cyclic
subgroup $C_{j,j+1}$ of $G$ such that $\{u_j, u_{j+1}\} \subseteq C_{j,j+1}$, for every $1 \leq j \leq n - 1$. Moreover, we have
$C_{j,j+1} \cap T_j \neq e$ and $C_{j,j+1} \cap T_{j+1} \neq e$, which implies that $T_j \sim C_{j,j+1}$ and $C_{j,j+1} \sim T_{j+1}$ in
$\Gamma_{MC}(G)$. Consequently, we have:

$$C = T_1 \sim C_{1,2} \sim T_2 \sim C_{2,3} \sim \ldots \sim T_{n-1} \sim C_{n-1,n} \sim T_n = T,$$

which is a path between $C$ and $T$ in $\Gamma_{MC}(G)$. It follows that $\Gamma_{MC}(G)$ is connected.

Now let $\Gamma_{MC}(G)$ be connected and $x, y \in G^\ast$. We consider that $C$ and $T$ are two maximal
cyclic subgroups of $G$ such that $x \in C$ and $y \in T$. We know that there exists a path between $C$
and $T$, so we have:

$$C = C_0 \sim C_1 \sim C_2 \sim \ldots \sim C_{n-1} \sim C_n = T.$$

Let $C_i = \langle z_i \rangle$, for every $0 \leq i \leq n$ and $e \neq w_{j,j+1} \in C_j \cap C_{j+1}$, for every $0 \leq j \leq n - 1$. We
have $z_j, w_{j,j+1} \in C_j$ and $w_{j,j+1}, z_{j+1} \in C_{j+1}$, which implies that $z_j \sim w_{j,j+1}$ and $w_{j,j+1} \sim z_{j+1}$,
respectively, in $\Gamma^{\ast}_{EP}(G)$, for every $0 \leq j \leq n - 1$. In addition, $x, z_0 \in C_0$ and $y, z_n \in C_n$ so
$x \sim z_0$ and $z_n \sim y$, in $\Gamma^{\ast}_{EP}(G)$. Consequently, we have the following path between $x$ and $y$ in
$\Gamma^{\ast}_{EP}(G)$:

$$x \sim z_0 \sim w_{0,1} \sim z_1 \sim \ldots \sim w_{n-1,n} \sim z_n \sim y,$$

which implies that $\Gamma^{\ast}_{EP}(G)$ is connected.

\[\square\]

**Theorem 2.2.** Let $G$ be a finite group. If $\Gamma_{MC}(G)$ is connected, then $\Gamma_{\mu}(G)$ is connected.

**Proof.** Let $\Gamma_{MC}(G)$ be connected. Assume that $s, t \in \mu(G)$, which implies that there are maximal
cyclic subgroups $S$ and $T$ of $G$ such that $|S| = s$ and $|T| = t$. Since $\Gamma_{MC}(G)$ is connected,
so there exists a path between $S$ and $T$, hence we have:

$$S = C_0 \sim C_1 \sim \ldots \sim C_{n-1} \sim C_n = T,$$

where $C_i$ is a maximal cyclic subgroup of $G$, for every $0 \leq i \leq n$. Let $|C_i| = k_i$, for every
$0 \leq i \leq n$. According to definition of $\Gamma_{MC}(G)$ and since $C_i \sim C_{i+1}$, so $C_i \cap C_{i+1} \neq e$, for every
$0 \leq i \leq n - 1$. It follows that $|C_i| = |C_{i+1}|$, hence $(k_i, k_{i+1}) \neq 1$, for every $0 \leq i \leq n - 1$.
Since $C_i$ is a maximal cyclic subgroup of $G$, so $k_i \in \mu(G)$, for every $0 \leq i \leq n$. Therefore, by
definition of $\Gamma_{\mu}(G)$, there exists the following path between $s$ and $t$:

$$s = k_0 \sim k_1 \sim \ldots \sim k_{n-1} \sim k_n = t,$$
which implies that $\Gamma_\mu(G)$ is connected.

\begin{proof}
By Theorems 2.1 and 2.2, the proof is clear.
\end{proof}

**Corollary 2.3.** Let $G$ be a finite group. If $\Gamma_\mu(G)$ is disconnected, then $\Gamma_{EP^*}(G)$ is disconnected.

**Theorem 2.4.** Let $G$ be a finite group. Then $\Gamma_\mu(G)$ is connected if and only if $\Gamma_{GK}(G)$ is connected.

**Proof.** First, let $\Gamma_\mu(G)$ be connected and $p,q \in \pi(G)$. Hence there are $m,n \in \mu(G)$ such that $p \mid m$ and $q \mid n$. Also, we know that there exists a path between $m$ and $n$ in $\Gamma_\mu(G)$, so we have:

$$m = k_0 \sim k_1 \sim \ldots \sim k_{t-1} \sim k_t = n,$$

where $k_i \in \mu(G)$, for every $0 \leq i \leq t$. Since $k_i \sim k_{i+1}$, so $(k_i,k_{i+1}) \neq 1$, for every $0 \leq i \leq t - 1$.

Let $r_i$ be a prime divisor of $(k_i,k_{i+1}) \neq 1$, for every $0 \leq i \leq t - 1$. Therefore, we have $r_i \sim r_{i+1}$ is a divisor of $k_{i+1}$, for every $0 \leq i \leq t - 1$. It follows that $r_i \sim r_{i+1}$ in $\Gamma_{GK}(G)$, for every $0 \leq i \leq t - 1$. On the other hand, we know that $r_0,p \mid k_0$, so $p \sim r_0$ in $\Gamma_{GK}(G)$ and similarly we get that $r_t \sim q$ in $\Gamma_{GK}(G)$. Therefore, we have the following path between $p$ and $q$ in $\Gamma_{GK}(G)$:

$$p \sim r_0 \sim r_1 \sim \ldots \sim r_{t-1} \sim r_t \sim q,$$

which implies that $\Gamma_{GK}(G)$ is connected.

Conversely, let $\Gamma_{GK}(G)$ be connected and $m,n \in \mu(G)$. Let $p \in \pi(m)$ and $q \in \pi(n)$. Since $\Gamma_{GK}(G)$ is connected, so there exists a path between $p$ and $q$ in $\Gamma_{GK}(G)$, so we have:

$$p = r_0 \sim r_1 \sim \ldots \sim r_{t-1} \sim r_t = q,$$

where $r_i$ is a prime divisor of $|G|$, for every $0 \leq i \leq t$. Since $r_i \sim r_{i+1}$ in $\Gamma_{GK}(G)$, so there exists a number $u_{i,i+1} \in \mu(G)$ such that $r_i \sim u_{i,i+1}$, for every $0 \leq i \leq t - 1$. It follows that $r_i \sim (u_{i,i+1},u_{i+1,i+2})$, which implies that $u_{i,i+1} \sim u_{i+1,i+2}$ in $\Gamma_\mu(G)$, for every $0 \leq i \leq t - 2$.

On the other hand, we know that $p \mid (m,u_{0,1})$, so $m \sim u_{0,1}$ in $\Gamma_\mu(G)$. Similarly, we get that $u_{t-1,t} \sim n$ in $\Gamma_\mu(G)$. Consequently, we have the following path in $\Gamma_\mu(G)$:

$$m \sim u_{0,1} \sim u_{1,2} \sim \ldots \sim u_{t-1,t} \sim n.$$ 

It follows that $\Gamma_\mu(G)$ is connected.

\begin{proof}
By Theorems 2.1 and 2.2, the proof is clear.
\end{proof}

**Theorem 2.5.** Let $G$ be a finite group. If $\Gamma_{GK}(G)$ is disconnected, then $\Gamma_{EP}(G)$ is disconnected.

\begin{proof}
By Theorems 2.1 and 2.2, the proof is clear.
\end{proof}
Proof. It immediately comes from Theorem 2.4 and Corollary 2.3.

Corollary 2.6. Let $G$ be a finite group. If $\Gamma_{\mathcal{G}K}(G)$ is disconnected, then $\Gamma_p^*(G)$ is disconnected.

Proof. Since $\Gamma_p^*(G)$ is a subgraph of $\Gamma_{\mathcal{E}P}^*(G)$, by Theorem 2.5, it is clear.

Corollary 2.7. If $G$ is isomorphic to one of the following groups, then $\Gamma_p^*(G)$ and $\Gamma_{\mathcal{E}P}^*(G)$ are disconnected:

1) A finite simple group whose prime graph is disconnected,
2) The dihedral group $D_{2n}$ where $n$ is an odd number.
3) The symmetric groups $S_p$, $S_{p+1}$, $S_{p+2}$.

Proof. By [9] and the structure of the above groups, we know that the prime graph of these groups are disconnected. So by Theorem 2.5 and Corollary 2.6, we get the result.

3 Connectivity of proper enhanced power graph

Lemma 3.1. Let $G$ be a finite group and $x, y \in G^*$ such that $xy = yx$. If $o(x) \neq o(y)$, then there exists a path from $x$ to $y$ in $\Gamma_{\mathcal{E}P}^*(G)$.

Proof. Let $d = (o(x), o(y))$. We consider the following cases:

Case 1. Let $d = 1$. Since $x$ and $y$ commute, we conclude that $x$ and $y$ belong to the cyclic subgroup $\langle xy \rangle$. According to the definition of enhanced power graph, we get that $x \sim y$ in $\Gamma_{\mathcal{E}P}^*(G)$.

Case 2. Let $d \neq 1$. By the assumption, $o(x) \neq o(y)$. This implies that either $x^d \neq e$ or $y^d \neq e$. If $x^d \neq e$, then $(o(x^d), o(y)) = 1$ and so similar to Case 1, $x \sim x^d \sim y$ in $\Gamma_{\mathcal{E}P}^*(G)$. If $y^d \neq e$, then $(o(y^d), o(x)) = 1$ and again we get that, $x \sim y^d \sim y$ in $\Gamma_{\mathcal{E}P}^*(G)$. This shows that in every case, there is a path between $x$ and $y$ in $\Gamma_{\mathcal{E}P}^*(G)$, which completes the proof.

We note that the condition $o(x) \neq o(y)$, in the previous lemma, is necessary. For instance, let $G = Z_p \times Z_p$. If $x, y \in G^*$, then $o(x) = o(y)$. Also, the proper enhanced power graph of $G$ is not connected.

Theorem 3.2. Let $G$ be a finite group. If $Z(G)$ is not an elementary abelian $p$-group, where $p$ is a prime number, then $\Gamma_{\mathcal{E}P}^*(G)$ is connected.
Proof. By the assumption, we may assume that there exist \( z_1, z_2 \in Z(G) \setminus \{ e \} \), such that \( o(z_1) \neq o(z_2) \). Hence, since \( z_1, z_2 \in Z(G) \), so \( z_1 \sim z_2 \) in \( \Gamma^*_E(P)(G) \), by Lemma 3.1. Moreover, let \( x \) be an arbitrary element of \( G \setminus \{ e, z_1, z_2 \} \). Since \( o(z_1) \neq o(z_2) \), we get that \( o(x) \neq o(z_1) \) or \( o(x) \neq o(z_2) \). Again, by Lemma 3.1, we deduce that there is a path between \( x \) and \( z_1 \) or \( z_2 \). Therefore, by this argument, \( \Gamma^*_E(P)(G) \) is connected.

Corollary 3.3. If \( G \) is a finite group such that \( |\pi(Z(G))| \geq 2 \), then \( \Gamma^*_E(P)(G) \) is connected.

Proof. We know that if \( |\pi(Z(G))| \geq 2 \), then \( Z(G) \) is not an elementary abelian \( p \)-group. So by Theorem 3.2, we get the result.

Corollary 3.4. If \( G \) is a finite nilpotent group which is not a \( p \)-group, then \( \Gamma^*_E(P)(G) \) is connected.

Proof. If \( G \) is a nilpotent group, then \( G = P_1 \times P_2 \times \cdots \times P_k \) and \( Z(G) = Z(P_1) \times Z(P_2) \times \cdots \times Z(P_k) \), where \( P_i \)'s are the Sylow subgroups of \( G \). Also, by the assumption, \( G \) is not a \( p \)-group. It follows that \( k \geq 2 \) and so \( |\pi(Z(G))| \geq 2 \). Therefore, by Corollary 3.3, \( \Gamma^*_E(P)(G) \) is connected.

Since every abelian group is a nilpotent group, so we have the following result:

Corollary 3.5. Proper enhanced power group of every abelian group which is not a \( p \)-group is connected.

References


