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THREE SOLUTIONS FOR A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. The purpose of this work is to study the following elliptic problem:

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta u = f(x)|u(x)|^{p-2}u(x) + \lambda g(x)|u|^{q-2}u \quad \text{in } \Omega; \\ u = 0, \quad \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, be a bounded smooth domain, f, $g \in L^\infty(\Omega)$, λ is a positive parameter. Under adequate assumptions on the sources terms f and g we establish the existence of three solutions, one of these is positive, one negative and the other change sign solution.
Keywords: The Laplacian operator, elliptic problem, Nehari manifold, three critical points, weak solution.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$. In this work, we show the existence of at least three solutions for the semilinear elliptic boundary-value problem:

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta u = f(x)|u(x)|^{p-2}u(x) + \lambda g(x)|u|^{q-2}u \quad \text{in } \Omega, \\ \\ u = 0, \quad \text{in } \partial\Omega, \end{cases}$$

where $f, g \in L^\infty(\Omega)$, λ is a positive parameter, $2 < p < q$ and we shall work in the space $H_0^1(\Omega)$ which can be equipped with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Such problems arise, for instance, in models of pseudo-plastic flows, reaction diffusion problems, nonlinear elasticity (see [2, 4] with references therein); the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [6] with references therein).

The number of the solutions for different kinds of Dirichlet problem has been widely investigated in literature , see for instance [1, 3, 9, 11, 12, 13, 14,

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16] and references therein. The approach in this work follows the one in [15].

It is clear that problem (P_λ) has a variational structure. We understand critical points of the associated energy functional acting on the Sobolev space $H_0^1(\Omega)$:

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega f(x)|u(x)|^p dx - \frac{\lambda}{q}\int_\Omega g(x)|u(x)|^q dx \tag{1.1}$$

Definition 1.1. *$u \in H_0^1(\Omega)$ is called a weak solution (solutions, for short) of problem (P_λ), that is, for functions $u \in H_0^1(\Omega)$ satisfying $\operatorname{ess\,inf}_K |u| > 0$ over every compact set $K \subset \Omega$ and*

$$\int_\Omega \nabla u \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega f(x)|u(x)|^{p-2}u(x)\, \mathrm{d}x + \lambda \int_\Omega g(x)|u(x)|^{q-2}u(x)\phi \, \mathrm{d}x \tag{1.2}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ with compact support.

Remark 1.1. *Obviously, it is easy to see that $\mathcal{J}_\lambda(u) \in C^1(H_0^1(\Omega), \mathbb{R})$. Moreover for all $u, v \in H_0^1(\Omega)$, we have*

$$\mathcal{J}_\lambda'(u)v = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega f(x)|u|^{p-2}u\, v\, dx - \lambda \int_\Omega g(x)|u|^{q-2}u\, v\, dx.$$

So, every weak solution of problem (P_λ) is also a solution of equation (1.2).

Before stating our main results, we make the following assumptions on the source terms f and g throughout this paper:

- **(F)** $f : \Omega \to \mathbb{R}$ is continuous such that there exist positive suitable constants $c_1\, C_1$ with $c_1 \leq f(x) \leq C_1$.
- **(G)** $g : \Omega \to \mathbb{R}$ is continuous such that there exist positive suitable constants $c_2\, C_2$ with $c_2 \leq g(x) \leq C_2$ for all $x \in \Omega$

Here we state our main result asserted in the following theorem.

Theorem 1.1. *Under the assumptions (**F**) and (**G**), there exists λ_0 such that for every $\lambda > \lambda_0$, problem (P_λ) has at least three, nontrivial, (weak) solutions. Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

2. Proof of our main results

In our proof, we adopt the approach used in [15]. That is, we will construct three disjoint sets $\mathcal{N}_i \neq \emptyset$, $i = 1, 2, 3$ not containing 0 such that $\mathcal{J}_\lambda$ has a critical point in $\mathcal{N}_i$. In fact, let

$$\mathcal{M}_1 = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^+ dx > 0,\, ||u^+||^2 - \int_\Omega f(x)|u^+|^p dx = \lambda \int_\Omega g(x)|u^+|^q dx \right\},$$

$$\mathcal{M}_2 = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^- dx > 0,\, ||u^-||^2 - \int_\Omega f(x)|u^-|^p dx = \lambda \int_\Omega g(x)|u^-|^p dx \right\},$$

and

$$\mathcal{M}_3 = \mathcal{M}_1 \cap \mathcal{M}_2.$$

As usual, $u^+ = \max(u, 0)$ and $u- = \max(-u, 0)$. Next, we define the sets $\mathcal{N}_i$ as follows

$$\mathcal{N}_1 = \{u \in \mathcal{M}_1 : u \geq 0\},$$
$$\mathcal{N}_2 = \{u \in \mathcal{M}_2 : u \leq 0\},$$

and

$$\mathcal{N}_3 = \mathcal{M}_3.$$

To proof our results, we need the following lemmas.

Lemma 2.1. *For every $\varphi \in H_0^1(\Omega)$ with $\varphi > 0$ ($\varphi < 0$), there exists $T_\lambda > 0$ such that $T_\lambda \varphi \in \mathcal{M}_1$ ($T_\lambda \varphi \in \mathcal{M}_2$). Moreover, we have $\lim\limits_{\lambda \to \infty} T_\lambda = 0$.*

Proof. Firstly,let us define the functional $\mathcal{E}_\lambda$ on $H_0^1(\Omega)$ as follows

$$\mathcal{E}_\lambda(u) = \|u\|^2 - \int_\Omega f(x)|u|^p dx - \lambda \int_\Omega g(x)|u|^q dx.$$

Then, for all $\varphi \in H_0^1(\Omega)$ with $\varphi > 0$, we obtain

$$\begin{aligned} \mathcal{E}_\lambda(t\varphi) &= t^2\|\varphi\|^2 - t^p \int_\Omega f(x)|\varphi|^p dx - \lambda t^q \int_\Omega g(x)|\varphi|^q dx \\ &\geq t^2\|\varphi\|^2 - C_1 t^p \int_\Omega |\varphi|^p dx - \lambda C_2 t^q \int_\Omega |\varphi|^r dx \\ &\geq t^2 \left(\|\varphi\|^2 - C_1 t^{p-2} \int_\Omega |\varphi|^p dx - \lambda C_2 t^{q-2} \int_\Omega |\varphi|^q dx \right). \end{aligned}$$

Since $2 < q < p$, it follows that $\mathcal{E}_\lambda(t\varphi) > 0$ for t small enough.
On the other hand

$$\begin{aligned} \mathcal{E}_\lambda(t\varphi) &= t^2\|\varphi\|^2 - t^p \int_\Omega f(x)|\varphi|^p dx - \lambda t^q \int_\Omega g(x)|\varphi|^q dx \\ &\leq t^2\|\varphi\|^2 - c_1 t^p \int_\Omega |\varphi|^p dx - \lambda c_2 t^q \int_\Omega |\varphi|^q dx \\ &\leq t^p \left(\frac{\|\varphi\|^2}{t^{p-2}} - c_1 \int_\Omega |\varphi|^p dx - \frac{c_2 \lambda}{t^{p-q}} \int_\Omega |\varphi|^q dx \right). \end{aligned}$$

Since $2 < q < p$, it follows that $\mathcal{E}_\lambda(t\varphi) < 0$ for t large enough. Consequently, by Bolzano's Theorem, there exists T_λ such that $\mathcal{E}_\lambda(T_\lambda \varphi) = 0$. Now, let us prove that $\lim\limits_{\lambda \to \infty} T_\lambda = 0$. Put

$$t_1 = \left(\frac{\|\varphi\|^2}{\lambda \int_\Omega g(x)|\varphi(x)|^q dx} \right)^{\frac{1}{q-2}}.$$

So

$$\mathcal{E}_\lambda(t_1\varphi) = -t_1^p \int_\Omega g(x)|\varphi(x)|^q dx < 0.$$

Therefore, we can choose T_λ such that $0 < T_\lambda < t_1$ which implies that $T_\lambda \to 0$ as $\lambda \to \infty$.
The other case being similar. This completes the proof of the Lemma 2.1. □

Lemma 2.2. *The following properties of the manifolds $\mathcal{M}_i$ and $\mathcal{N}_i$ (i=1:3) holds*

(1) $\mathcal{M}_i$, $i = 1:3$ *is a $C^{1,1}$ sub-manifold of E. Moreover $\mathcal{M}_1$ and $\mathcal{M}_2$ are of codimension 1 while $\mathcal{M}_3$ is of codimension 2.*

(2) *The sets $\mathcal{N}_i$ (i=1:3) are complete. Moreover, for every $u \in \mathcal{M}_i, (i = 1:3)$ we have the direct decomposition*

$$T_uE = T_u\mathcal{M}_i \oplus span\{u_+, u_-\}, \tag{2.1}$$

where T_uF is the tangent space in $H^1(\Omega)$ at the point u of the Banach manifold F.

(3) *The projection onto the first component in (2.1) is uniformly continuous on bounded sets of $\mathcal{M}_i$.*

Proof.

(1) Let us denote

$$\widetilde{\mathcal{M}}_1 = \left\{ u \in H_0^1(\Omega) : \int_\Omega u_+ dx > 0 \right\},$$

$$\widetilde{\mathcal{M}}_2 = \left\{ u \in H_0^1(\Omega) : \int_\Omega u_- dx > 0 \right\},$$

$$\widetilde{\mathcal{M}}_3 = \widetilde{\mathcal{M}}_1 \cap \widetilde{\mathcal{M}}_2.$$

Obviously, $\mathcal{M}_i \subset \widetilde{\mathcal{M}}_i$. Using the Sobolev trace Theorem, the set $\widetilde{\mathcal{M}}_i$ is open in $H^1(\Omega)$. So, let us prove that $\mathcal{M}_i$ is a smooth sub-manifold of $\widetilde{\mathcal{M}}_i$. In order to do this, we will define the functional J_i on $\widetilde{\mathcal{M}}_i$ as follows:

$$J_1 : \widetilde{\mathcal{M}}_1 \to \mathbb{R}, \quad u \longmapsto \int_\Omega |\nabla u^+|^2 dx - \int_\Omega f(x)|u^+|^p dx - \lambda \int_\Omega g(x)|u^+|^q dx,$$

$$J_2 : \widetilde{\mathcal{M}}_2 \to \mathbb{R}, \quad u \longmapsto \int_\Omega |\nabla u^-|^2 dx - \int_\Omega f(x)|u^-|^p dx - \lambda \int_\Omega g(x)|u^-|^q dx,$$

and

$$J_3 : \widetilde{\mathcal{M}}_3 \longrightarrow \mathbb{R}^2, \quad u \longmapsto (J_1(u), J_2(u)).$$

It is easily, to see that for $i = 1:3$, we have $\mathcal{M}_i = J_i^{-1}(0)$. On the other hand, for all $u \in \mathcal{M}_1$, we have

$$\begin{aligned} \langle \nabla J_1(u), u^+ \rangle &= 2\|u^+\|^2 - p\int_\Omega f(x)|u^+(x)|^p dx - \lambda q \int_\Omega g(x)|u^+|^p dx \\ &\leq 2\left(\|u^+\|^2 - \int_\Omega f(x)|u^+|^p dx \right) - \lambda q \int_\Omega g(x)|u^+(x)|^p dx \end{aligned}$$

$$= \quad \lambda(2-q)\int_\Omega g(x)|u^+|^p dx < 0.$$

This implies that 0 is a regular value of J_1, that is $\mathcal{M}_1$ is a $C^{1,1}$ manifold of codimension 1. The same argument that $\mathcal{M}_2$ is a $C^{1,1}$-manifold of codimension 1 and $\mathcal{M}_3$ is a $C^{1,1}$-manifold of codimension 2.

(2) Obviously, we have that

$$\mathcal{M}_1 = \left\{ u \in \widetilde{\mathcal{M}}_1 : J_1(u) = 0 \right\}$$

and

$$T_u\mathcal{M}_1 = \{ v : \langle \nabla J_1(u), v\rangle = 0 \}.$$

Since, for all $u \in \mathcal{M}_1$, $\langle \nabla J_1(u), u_+\rangle < 0$, then, one has

$$T_u\mathcal{M}_1 + span(u^+) = T_u\mathcal{M}_1 \oplus span(u^+) \subset T_u(H_0^1(\Omega)).$$

Fix $u \in \mathcal{M}_1$, and let $v \in E$, then for

$$\alpha := \frac{\langle \nabla J_1(u), v\rangle}{\langle \nabla J_1(u), u^+\rangle},$$

we have

$$v = v_1 + v_2 \ \text{ with } v_1 = \alpha u^+ \in span(u^+) \ \text{ and } \ v_2 = v - v_1.$$

Moreover $\langle \nabla J_1(u), v_2\rangle = \langle \nabla J_1(u), v\rangle - \langle \nabla J_1(u), v_1\rangle = 0$, which implies that $v_2 \in T_u\mathcal{M}_1$.
The same argument can be applied to prove that $T_u(H_0^1(\Omega)) = T_u\mathcal{M}_2 \oplus span\{u^-\}$ and $T_u(H_0^1(\Omega)) = T_u\mathcal{M}_3 \oplus span\{u^+, u^-\}$.

(3) The uniform continuity of the projections onto $T_u\mathcal{M}_i$ follows immediately from (2.1) and the estimates given in the proof of part **(1)**.

This completes the proof of the Lemma 2.2. □

Now, we define a generalized notion of Palais-Smale sequence for $\mathcal{J}_\lambda$.

Definition 2.1. *The functional $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition for energy level c, if all sequence $\{u_k\}$ such that*

$$\mathcal{J}_\lambda(u_k) \to c \ \text{ and } \ \mathcal{J}_\lambda'(u_k) \to 0$$

admits a convergent sub-sequence.

We recall the following result due to GARCIA AZORERO-PERAL [8].

Lemma 2.3. *Let S be the best Sobolev constant*

$$S := \inf_{v \in C_c^\infty(\Omega)} = \frac{\int_\Omega |\nabla v|^2 dx}{\left(\int_\Omega |v|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

Then, for all $c < \frac{1}{N} S^{\frac{N}{2}}$, the unrestricted functional $\mathcal{J}_\lambda$ verifies the Palais-Smale condition for energy level c.

Now, we need to check the Palais-Smale condition for the functional $\mathcal{J}_{\lambda|_{K_i}}$.

Lemma 2.4. *For all $c < \frac{1}{N}S^{\frac{N}{2}}$, the functional $\mathcal{J}_{\lambda|_{K_i}}$, satisfies the Palais-Smale condition for energy level c.*

Proof. Put $c < \frac{1}{N}S^{\frac{N}{2}}$ and let $\{u_k\} \subset K_i$ be a Palais-Smale sequence for a level c, that is

$$\mathcal{J}_\lambda(u_k) \to c \text{ and } \mathcal{J}_\lambda'(u_k) \to 0.$$

We need to show that there exists a subsequence denoted also $\{u_k\}$ that converge strongly in K_i.

Now, let $v_k \in T_{u_k}(H_0^1(\Omega))$ be a unit tangential vector such that

$$\langle \nabla \mathcal{J}_\lambda(u_k, v_k)\rangle = \|\nabla \mathcal{J}_\lambda(u_k\|_{(H_0^1(\Omega))^{-1}}.$$

From Lemma 2.2 (ii), there exist $x_k \in T_{u_k}\mathcal{M}_i$ and $y_k \in span\{(u_k)^+, (u_k)^-\}$ such that $v_k = x_k + y_k$. On the other hand, by the Sobolev embedding, there exists $\alpha, \beta > 0$ such that for all $u \in \mathcal{N}_i$ we have

$$\alpha\|u\|^2 \leq \mathcal{J}_\lambda(u) \leq \beta\|u\|^2. \tag{2.2}$$

Now, since $\mathcal{J}_\lambda(u_k)$ is uniformly bounded and using (2.2), u_k is uniformly bounded in $H_0^1(\Omega)$ and hence x_k is uniformly bounded in $H_0^1(\Omega)$. Therefore,

$$\|\nabla \mathcal{J}_\lambda(u_k\|_{(H_0^1(\Omega))^{-1}} = \langle \nabla \mathcal{J}_\lambda(u_k, v_k)\rangle = \langle \nabla \mathcal{J}_{\lambda|_{\mathcal{N}_i}}(u_k, v_k)\rangle \to 0. \tag{2.3}$$

As x_k is uniformly bounded and $\mathcal{J}_{\lambda|_{\mathcal{N}_i}}(u_k) \to 0$ strongly, then we have convergence strongly to 0 in (2.3) and the result follows from Lemma 2.3. This the proof of the Lemma 2.4. □

As a consequence of the Lemma 2.4 we have immediately the following result.

Lemma 2.5. *Let $u \in \mathcal{N}_i$ be a critical point of the restricted functional $\mathcal{J}_{\lambda|_{\mathcal{N}_i}}$. Then, u is also a critical point of the functional $\mathcal{J}_\lambda$ and so a weak solution to problem* (P_λ).

Proof of Theorem 1.1. Since $\mathcal{J}_\lambda$ is bounded below on $\mathcal{N}_1$. Therefore, by Ekeland's Variational Principle, there exists $u_k \in \mathcal{N}_1$ such that

$$\mathcal{J}_\lambda(u_k) \to c := \inf_{v\in\mathcal{N}_1} \mathcal{J}_\lambda(v) \text{ and } \left(\mathcal{J}_{\lambda|_{\mathcal{N}_1}}\right)'(u_k) \to 0.$$

Let $\omega \geq 0$, then, it follows from Lemma 2.1 that there exists $T_\lambda > 0$ such that

$$c \leq \mathcal{J}_\lambda(t_\lambda\omega) \leq \frac{1}{2}T_\lambda^2\|\omega\|^2.$$

Since $T_\lambda \to 0$, then, $c \to 0$ as $\lambda \to \infty$. This implies that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ we have $c < \frac{1}{N}S^{\frac{N}{2}}$. Combining this with Lemma 2.2 Lemma 2.4 and Lemma 2.5 we obtain a weak solution to problem (P_λ) in $\mathcal{N}_1$. The other cases are similar. Finally, by construction of $\mathcal{N}_i$ ($i = 1 : 3$), one of this solutions is positive, one is negative and the other changes sign. This completes the proof of the Theorem 1.1. □

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