ON ARBITRARILY GRADED RINGS

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ABSTRACT. Let $\mathfrak{R}$ be a ring graded by an arbitrary set A. We show that $\mathfrak{R}$ decomposes as the sum of well-described graded ideals plus (maybe) a certain subgroup. We also provide a context where the graded simplicity of $\mathfrak{R}$ is characterized and where a second Wedderburn-type theorem in the category of arbitrarily graded rings is stated.

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1. INTRODUCTION AND PREVIOUS DEFINITIONS

The interest on group-gradings on (associative) rings has been remarkable in the last years (see for instance [3, 7, 12, 13, 15, 18, 19]). We can also consider the recent monograph [11] and the survey [14]. Also group-gradings on Lie algebras has attracted the interest of many authors (see [2, 8, 9, 10, 17]). However gradings by means of an arbitrary set, not necessarily a group, have been barely considered in the literature. The present paper is devoted to the study of arbitrary rings graded through arbitrary sets.

Definition 1.1. Let $\mathfrak{R}$ be an (associative) ring and $A \neq \emptyset$ an arbitrary set. It is said that $\mathfrak{R}$ is a *graded ring*, by means of A, if

$$\mathfrak{R} = \bigoplus_{i \in A} \mathfrak{R}_i$$

where any $\mathfrak{R}_i$ is a subgroup of $\mathfrak{R}$, called *homogeneous component*, satisfying that for any $i, j \in A$ either $\mathfrak{R}_i\mathfrak{R}_j = \{0\}$ or $\{0\} \neq \mathfrak{R}_i\mathfrak{R}_j \subset \mathfrak{R}_k$ for some (unique) $k \in A$.

The *support* of the grading is the set

$$\Sigma := \{i \in A : \mathfrak{R}_i \neq \{0\}\}.$$

As classical examples of graded rings we have the group-graded rings, (see the hand book [16]) or the Peirce decomposition of an associative algebra respect to a family of commuting idempotents. In order to provide some detailed examples we introduce the next ring:

Let I and J be two arbitrary nonempty sets and $\mathcal{R}$ an arbitrary ring with 1. Consider the set

$$\mathfrak{S} := \mathcal{R}^{(I\times J)\times(I\times J)}$$

of all $\mathcal{R}$-valued mappings a on $(I \times J) \times (I \times J)$ with just a finite number of non-zero values and such that

$$a((i, j), (l, m)) = 0$$

when $j \neq m$.

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The latter, endowed with "point-wise" sum becomes an Abelian group and a ring with "matrix" multiplication

$$(ab)((i,j),(l,m)) = \sum_{(k,s)\in I\times J} a((i,j),(k,s))b((k,s),(l,m)),$$

for all $a, b \in \mathcal{R}^{(I\times J)\times(I\times J)}$.

For any $((i,j),(k,j))$, $i,k \in I$ and $j \in J$, we will denote by

$$E_{((i,j),(k,j))} : (I\times J)\times(I\times J) \to \mathcal{R}$$

the element in $\mathfrak{S}$ given by

$$E_{((i,j),(k,j))}((l,m),(n,s)) := \begin{cases} 1, & \text{if } ((l,m),(n,s)) = ((i,j),(k,j)); \\ 0, & \text{otherwise.} \end{cases}$$

Let us present several gradings on $\mathfrak{S}$.

Example 1.1. For any $(i,j,k) \in I\times I\times J$ denote by

$$\mathfrak{S}_{(i,j,k)} := \mathcal{R}E_{((i,k),(j,k))},$$

where for any $r \in \mathcal{R}$ the map $rE_{((i,k),(j,k))}$ denotes

$$(rE_{((i,j),(k,j))})((l,m),(n,s)) := \begin{cases} r, & \text{if } ((l,m),(n,s)) = ((i,j),(k,j)); \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathfrak{S}$ clearly admits an $(I\times I\times J)$-grading given by

$$\mathfrak{S} = \bigoplus_{(i,j,k)\in I\times I\times J} \mathfrak{S}_{(i,j,k)}.$$

Example 1.2. Let us fix an arbitrary abelian group G. We have that any function

$$\phi : I\times J \to G$$

gives rise to a G-grading on $\mathfrak{S}$ given by

$$\mathcal{R}E_{((i,j),(k,j))} \subset \mathfrak{S}_g \text{ if and only if } g = \phi(i,j)^{-1}\phi(k,j).$$

Indeed, taking into account $E_{((i,j),(k,j))}E_{((m,l),(n,l))} = 0$ for $(k,j) \neq (m,l)$, and

$$\phi(i,j)^{-1}\phi(k,j)\phi(k,j)^{-1}\phi(n,j) = \phi(i,j)^{-1}\phi(n,j),$$

the above condition clearly defines the grading

$$\mathfrak{S} = \bigoplus_{g\in G} \mathfrak{S}_g$$

with

$$\mathfrak{S}_g = \bigoplus \mathcal{R}E_{((i,j),(k,j))}, \tag{1}$$

where the direct sum is taken over all $i,k \in I; j \in J$ with

$$\phi(i,j)^{-1}\phi(k,j) = g.$$

A *graded subring* $\mathfrak{T}$ of an arbitrarily graded ring $\mathfrak{R} = \bigoplus_{i\in A} \mathfrak{R}_i$ is a subgroup of $\mathfrak{R}$ satisfying $\mathfrak{T}\mathfrak{T} \subset \mathfrak{T}$ and such that splits as $\mathfrak{T} = \bigoplus_{i\in A} \mathfrak{T}_i$ with any $\mathfrak{T}_i = \mathfrak{T} \cap \mathfrak{R}_i$. A graded subring $\mathfrak{I}$ of $\mathfrak{R}$ is a *graded ideal* if $\mathfrak{R}\mathfrak{I} + \mathfrak{I}\mathfrak{R} \subset \mathfrak{I}$. Finally, $\mathfrak{R}$ is called *graded simple* if its product is nonzero and its only graded ideals are $\{0\}$ and $\mathfrak{R}$.

The paper is organized as follows. In Chapter 2 we improve the connections techniques on the support of a grading, developed for group-graded associative algebras in [5], so as to get our tool for the study of arbitrary gradings on rings. These techniques will allow us to associate an adequate graded ideal to each equivalence class given by the connection relation on the support of the grading, (it turns out to be an equivalence relation), and show that any ring $\mathfrak{R}$ with an arbitrary grading decomposes as the sum of these well-described graded ideals plus (maybe) a certain subgroup.

In Chapter 3, and under mild conditions, the graded simplicity of $\mathfrak{R}$ is characterized and it is shown that the above decomposition is given by the family of the minimal graded ideals of $\mathfrak{R}$, (each one being a graded simple ring).

2. Connections in the support techniques. First results

We begin this section by developing the main tool in our study.

Let
$$\mathfrak{R} = \bigoplus_{i\in A} \mathfrak{R}_i$$
be a graded ring by means of the non-empty set A. By renaming if necessary, we will suppose $\emptyset \notin A$ and we will denote by $\Sigma \subset A$ the support of the grading. For each $i \in \Sigma$, a new variable $\overline{i} \notin \Sigma$ is introduced and we denote the set consisting of all these new symbols by
$$\overline{\Sigma} := \{\overline{i} : i \in \Sigma\}.$$

Given $\overline{i} \in \overline{\Sigma}$ we will also denote $\overline{(\overline{i})} := i$. Now, for any subset $\mathfrak{A}$ of $\Sigma \,\dot\cup\, \overline{\Sigma}$ we write $\overline{\mathfrak{A}} := \{\overline{i} \in \Sigma \,\dot\cup\, \overline{\Sigma} : i \in \mathfrak{A}\}$ if $\mathfrak{A} \neq \emptyset$ and $\overline{\emptyset} = \emptyset$.

By denoting $\mathcal{P}(B)$ to the power set of a given set B, we introduce the mapping $\star$ which recover certain multiplicative relations among the homogeneous components of the grading.
$$\star : (\Sigma \,\dot\cup\, \overline{\Sigma}) \times (\Sigma \,\dot\cup\, \overline{\Sigma}) \to \mathcal{P}(\Sigma),$$
defined by

- For $i, j \in \Sigma$,
$$i \star j = \begin{cases} \emptyset, & \text{if } \{0\} = \mathfrak{R}_i\mathfrak{R}_j; \\ \{k\}, & \text{if } \{0\} \neq \mathfrak{R}_i\mathfrak{R}_j \subset \mathfrak{R}_k. \end{cases}$$
- For $i \in \Sigma$ and $\overline{j} \in \overline{\Sigma}$,
$$i \star \overline{j} = \overline{j} \star i = \{k \in \Sigma : 0 \neq \mathfrak{R}_k\mathfrak{R}_j \subset \mathfrak{R}_i\} \cup \{l \in \Sigma : 0 \neq \mathfrak{R}_j\mathfrak{R}_l \subset \mathfrak{R}_i\}.$$
- For $\overline{i}, \overline{j} \in \overline{\Sigma}$,
$$\overline{i} \star \overline{j} = \emptyset.$$

The proof of the next result is immediate.

Lemma 2.1. *Let $k \in \Sigma$ be. Then the following assertions hold.*

(i) *For any $i, j \in \Sigma$, $k \in i \star j \cup j \star i$ if and only if $i \in k \star \overline{j}$.*
(ii) *For any $\overline{i} \in \overline{\Sigma}$ and $j \in \Sigma$, $k \in \overline{i} \star j$ if and only if $i \in j \star \overline{k}$.*

Now we have to note that it is interesting to distinguish one element $\mathfrak{o}$ in the support of the grading. This allows us to cover the cases in which there exists a homogeneous space $\mathfrak{R}_{\mathfrak{o}}$ which has a different behavior to the remaining homogeneous spaces. This is for instance the case in which the grading set A is an Abelian group, where the homogeneous space $\mathfrak{R}_0$ associated to the zero element 0 in the group enjoys of a distinguished role (see [5, 16]). From here, we are going to distinguish in our study one element $\mathfrak{o}$ in the support of the grading (satisfying an additional condition). Hence, let us fix an element $\mathfrak{o}$ such that either $\mathfrak{o} \in \Sigma$ and satisfies $\mathfrak{o} \star i \neq \{\mathfrak{o}\}$ and $i \star \mathfrak{o} \neq \{\mathfrak{o}\}$ for any $i \in \Sigma \setminus \{\mathfrak{o}\}$, or $\mathfrak{o} = \emptyset$. Denote also

$$\Delta := \Sigma \setminus \{\mathfrak{o}\} \text{ and } \overline{\Delta} := \overline{\Sigma} \setminus \{\overline{\mathfrak{o}}\}.$$

Note that the possibility $\mathfrak{o} = \emptyset$ holds for the case in which it is not wished to distinguish any element in Σ.

Example 2.1. In the $(I \times I \times J)$-grading given in Example 1.1 we do not wish to distinguish any element in the support. So $\mathfrak{o} = \emptyset$ and $\Sigma = \Delta$. However, in the G-grading given in Example 1.2 we want to distinguish the homogeneous component associated to the zero element of G. Then we will take $\mathfrak{o} = 0$ and $\Delta = \Sigma \setminus \{0\}$.

Finally, we introduce the map

$$\phi : \mathcal{P}(\Delta \dot\cup \overline{\Delta}) \times (\Sigma \dot\cup \overline{\Sigma}) \to \mathcal{P}(\Delta \dot\cup \overline{\Delta}),$$

as

- $\phi(\emptyset, a) = \emptyset$ for all $a \in \Sigma \dot\cup \overline{\Sigma}$.
- For any $\emptyset \neq \mathfrak{A} \in \mathcal{P}(\Delta \dot\cup \overline{\Delta})$ and $a \in \Sigma \dot\cup \overline{\Sigma}$,

$$\phi(\mathfrak{A}, a) = \Big(\Big(\bigcup_{x \in \mathfrak{A}} \{x \star a, a \star x\}\Big) \setminus \{\mathfrak{o}\}\Big) \cup \overline{\Big(\Big(\bigcup_{x \in \mathfrak{A}} \{x \star a, a \star x\}\Big) \setminus \{\mathfrak{o}\}\Big)}.$$

Note that for any $\mathfrak{A} \in \mathcal{P}(\Delta \dot\cup \overline{\Delta})$ and $a \in \Sigma \dot\cup \overline{\Sigma}$ we have that

$$\phi(\mathfrak{A}, a) = \overline{\phi(\mathfrak{A}, a)} \tag{2}$$

and

$$\phi(\mathfrak{A}, a) \cap \Delta = \Big(\bigcup_{x \in \mathfrak{A}} \{x \star a, a \star x\}\Big) \setminus \{\mathfrak{o}\}. \tag{3}$$

Lemma 2.2. *For any $\mathfrak{A} \in \mathcal{P}(\Delta \dot\cup \overline{\Delta})$ such that $\mathfrak{A} = \overline{\mathfrak{A}}$ and any $a \in \Sigma \dot\cup \overline{\Sigma}$ we have that $i \in \phi(\mathfrak{A}, a) \cap \Delta$ if and only if $i \in \Delta$ and either $\phi(\{i\}, \overline{a}) \cap \mathfrak{A} \cap \Delta \neq \emptyset$ or $\phi(\{\overline{i}\}, a) \cap \mathfrak{A} \cap \Delta \neq \emptyset$.*

Proof. It follows from Equation (3), Lemma 2.1 and the facts $j \star \overline{k} = \overline{j} \star k$, $\overline{j} \star \overline{k} = \emptyset$ for any $j, k \in \Sigma$. □

Definition 2.1. Let $i, j \in \Delta$. We say that i is *connected* to j if either $i = j$ or there exists a subset $\{a_1, a_2, ..., a_{n-1}, a_n\} \subset \Sigma \dot\cup \overline{\Sigma}$ with $n \geq 2$ such that the following conditions hold:

1. $a_1 \in \{i, \overline{i}\}$.

2. $\phi(\{a_1\}, a_2) \neq \emptyset$,
   $\phi(\phi(\{a_1\}, a_2), a_3) \neq \emptyset$,
   $\phi(\phi(\phi(\{a_1\}, a_2), a_3), a_4) \neq \emptyset$,
   $\vdots$
   $\phi(\phi(\cdots(\phi(\{a_1\}, a_2), \cdots), a_{n-2}), a_{n-1}) \neq \emptyset$.
3. $j \in \phi(\phi(\cdots(\phi(\{a_1\}, a_2), \cdots), a_{n-1}), a_n)$.

The subset $\{a_1, a_2, ..., a_{n-1}, a_n\}$ is called a *connection* from i to j.

Lemma 2.3. *Let $\{a_1, a_2, ..., a_{n-1}, a_n\}$ be, $n \geq 2$, a connection from i to j where $i, j \in \Delta$ with $i \neq j$. Then there exists a connection $\{j', a'_n, a'_{n-1}, \dots, a'_3, a'_2\}$ from j to i in such a way that $j' \in \{j, \overline{j}\}$ and $a'_i \in \{a_i, \overline{a_i}\}$ for $i \in \{2, ..., n\}$.*

Proof. Let us argue by induction on n.

If $n = 2$, then $a_1 \in \{i, \overline{i}\}$ and $j \in \Delta \cap \phi(\{a_1\}, a_2)$. From here

$$j \in (i \star a_2) \cup (a_2 \star i) \cup (\overline{i} \star a_2) \cup (a_2 \star \overline{i}).$$

Lemma 2.1 and the facts $j \star \overline{k} = \overline{j} \star k$, $\overline{j} \star \overline{k} = \emptyset$ for any $j, k \in \Sigma$ give us now

$$i \in \{j \star \overline{a}_2, \overline{j} \star a_2\}$$

and so we can find a set $\{j', a'_2\}$, with $j' \in \{j, \overline{j}\}$ and $a'_2 \in \{a_2, \overline{a_2}\}$, in such a way that it is a connection from j to i.

Suppose now the assertion holds for any connection with n, $n \geq 2$, elements and let us show the assertion also holds for any connection

$$\{a_1, a_2, ..., a_n, a_{n+1}\}$$

with $n + 1$ elements. From the facts

$$j \in \phi(\phi(\dots(\phi(\{a_1\}, a_2), \dots), a_n), a_{n+1}) \cap \Delta$$

and

$$\mathfrak{A} := \phi(\dots(\phi(\{a_1\}, a_2), \dots), a_n) = \overline{\phi(\dots(\phi(\{a_1\}, a_2), \dots), a_n)},$$

(see Equation (2)), Lemma 2.2 gives us either

$$\phi(\{j\}, \overline{a_{n+1}}) \cap \mathfrak{A} \cap \Delta \neq \emptyset$$

or

$$\phi(\{\overline{j}\}, a_{n+1}) \cap \mathfrak{A} \cap \Delta \neq \emptyset.$$

From here we can take some

$$k \in (\phi(\{j\}, \overline{a_{n+1}}) \cup \phi(\{\overline{j}\}, a_{n+1})) \cap \mathfrak{A} \cap \Delta$$

and so

$$k \in \phi(\{j'\}, a'_{n+1}) \tag{4}$$

for some $j' \in \{j, \overline{j}\}$ and $a'_{n+1} \in \{a_{n+1}, \overline{a_{n+1}}\}$.

On the other hand, the fact $k \in \mathfrak{A}$ allows us to assert that

$$\{a_1, a_2, ..., a_n\}$$

is a connection from i to k. By induction hypothesis we can take a connection

$$\{k', a'_n, a'_{n-1}, \dots, a'_3, a'_2\}$$

from k to i in such a way that $k' \in \{k, \overline{k}\}$ and $a'_i \in \{a_i, \overline{a_i}\}$ for $i \in \{2, ..., n\}$. From here, Equation (4) allows us to assert that

$$\{j', a'_{n+1}, a'_n, a'_{n-1}, \ldots, a'_3, a'_2\}$$

is a connection from j to i, which completes the proof. □

Proposition 2.1. *The relation $\sim$ in Δ, defined by $i \sim j$ if and only if i is connected to j, is an equivalence relation.*

Proof. The relation is reflexive by definition and symmetric by Lemma 2.3. Hence let us study the transitivity of $\sim$. Take $i, j, k \in \Delta$ such that $i \sim j$ and $j \sim k$. If $i = j$ or $j = k$ it is trivial, so suppose $i \neq j$ and $j \neq k$ and write $\{i_1, ..., i_n\}$ for a connection from i to j and $\{j_1, ..., j_m\}$ for a connection from j to k. Then we clearly have that $\{i_1, ..., i_n, j_2, ..., j_m\}$ is a connection from j to k. We have shown the connection relation is an equivalence relation. □

By the above Proposition we can consider the quotient set

$$\Delta/\sim = \{[i] : i \in \Delta\},$$

being $[i] = \{j \in \Delta : i \sim j\}$ the equivalence class of the element $i \in \Delta$.

Our next goal in this section is to associate a graded ideal $\mathfrak{R}_{[i]}$ of $\mathfrak{R}$ to any $[i]$. Fix $i \in \Delta$, we start by defining the subgroups $\mathfrak{R}_{\mathfrak{o},[i]} \subset \mathfrak{R}_{\mathfrak{o}}$ and $\mathfrak{V}_{[i]}$ as follows

$$\mathfrak{R}_{\mathfrak{o},[i]} := \Big(\sum_{j,k\in[i]} \mathfrak{R}_j \mathfrak{R}_k\Big) \cap \mathfrak{R}_{\mathfrak{o}} \text{ and } \mathfrak{V}_{[i]} := \bigoplus_{j\in[i]} \mathfrak{R}_j.$$

Finally, we denote by

$$\mathfrak{R}_{[i]} := \mathfrak{R}_{\mathfrak{o},[i]} \oplus \mathfrak{V}_{[i]}.$$

Also observe that we can write $\mathfrak{R} = \mathfrak{R}_{\mathfrak{o}} \oplus (\bigoplus_{[i]\in\Delta/\sim} \mathfrak{V}_{[i]})$.

Lemma 2.4. *For any $j \in [i]$ and any $k \in \Sigma$ we have $\mathfrak{R}_j\mathfrak{R}_k + \mathfrak{R}_k\mathfrak{R}_j \subset \mathfrak{R}_{[i]}$.*

Proof. The situation in which $\mathfrak{R}_j\mathfrak{R}_k + \mathfrak{R}_k\mathfrak{R}_j = \{0\}$ is immediate. Hence suppose either $\mathfrak{R}_j\mathfrak{R}_k \neq \{0\}$ or $\mathfrak{R}_k\mathfrak{R}_j \neq \{0\}$.

In the first case, there exists by the grading a unique $l \in \Sigma$ such that $\{0\} \neq \mathfrak{R}_j\mathfrak{R}_k \subset \mathfrak{R}_l$. Let us distinguish three possibilities. If $k \neq \mathfrak{o}$ and $l \neq \mathfrak{o}$ then the connection $\{j, k\}$ gives us $j \sim l$ and so $l \in [i]$. Hence $\mathfrak{R}_j\mathfrak{R}_k \subset \mathfrak{V}_{[i]} \subset \mathfrak{R}_{[i]}$. If $k \neq \mathfrak{o}$ and $l = \mathfrak{o}$ then $\{\overline{j}, \mathfrak{o}\}$ is a connection from j to k and then $k \in [i]$. From here $\mathfrak{R}_j\mathfrak{R}_k \subset \mathfrak{R}_{\mathfrak{o},[i]} \subset \mathfrak{R}_{[i]}$. Finally, if $k = \mathfrak{o}$ then necessarily $l \neq \mathfrak{o}$ and we have that the set $\{j, \mathfrak{o}\}$ shows $j \sim l$, therefore $l \in [i]$ and consequently $\mathfrak{R}_j\mathfrak{R}_k \subset \mathfrak{R}_l \subset \mathfrak{V}_{[i]} \subset \mathfrak{R}_{[i]}$.

In the second case we can argue in a similar way. □

Proposition 2.2. *For any $i \in \Delta$, the subgroup $\mathfrak{R}_{[i]}$ is a graded ideal of $\mathfrak{R}$.*

Proof. We can write

$$\mathfrak{R}\mathfrak{R}_{[i]} \subset \mathfrak{R}_{\mathfrak{o}}\mathfrak{R}_{\mathfrak{o},[i]} + \mathfrak{R}_{\mathfrak{o}}\mathfrak{V}_{[i]} + \Big(\bigoplus_{j\in\Delta} \mathfrak{R}_j\Big)\mathfrak{R}_{\mathfrak{o},[i]} + \Big(\bigoplus_{j\in\Delta} \mathfrak{R}_j\Big)\mathfrak{V}_{[i]} \tag{5}$$

and

$$\mathfrak{R}_{[i]}\mathfrak{R} \subset \mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{R}_{\mathfrak{o}} + \mathfrak{V}_{[i]}\mathfrak{R}_{\mathfrak{o}} + \mathfrak{R}_{\mathfrak{o},[i]}\Big(\bigoplus_{j\in\Delta} \mathfrak{R}_j\Big) + \mathfrak{V}_{[i]}\Big(\bigoplus_{j\in\Delta} \mathfrak{R}_j\Big). \tag{6}$$

Let $j,k \in [i]$ be, by Lemma 2.4 and the associativity of the product we have $\mathfrak{R}_{\mathfrak{o}}(\mathfrak{R}_k\mathfrak{R}_j) \subset \mathfrak{R}_{[i]}$ and $(\mathfrak{R}_k\mathfrak{R}_j)\mathfrak{R}_{\mathfrak{o}} \subset \mathfrak{R}_{[i]}$. From here

$$\mathfrak{R}_{\mathfrak{o}}\mathfrak{R}_{\mathfrak{o},[i]} + \mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{R}_{\mathfrak{o}} \subset \mathfrak{R}_{[i]}. \tag{7}$$

In a similar way, if $j,k \in [i]$ and $l \in \Delta$ we get $\mathfrak{R}_l(\mathfrak{R}_k\mathfrak{R}_j) + (\mathfrak{R}_k\mathfrak{R}_j)\mathfrak{R}_l \subset \mathfrak{R}_{[i]}$ and so

$$\Big(\bigoplus_{j\in\Delta}\mathfrak{R}_j\Big)\mathfrak{R}_{\mathfrak{o},[i]} + \mathfrak{R}_{\mathfrak{o},[i]}\Big(\bigoplus_{j\in\Delta}\mathfrak{R}_j\Big) \subset \mathfrak{R}_{[i]}. \tag{8}$$

Lemma 2.4 also gives us that $\mathfrak{R}_j\mathfrak{R}_{\mathfrak{o}} + \mathfrak{R}_{\mathfrak{o}}\mathfrak{R}_j \subset \mathfrak{R}_{[i]}$ for any $j \in [i]$ and so

$$\mathfrak{V}_{[i]}\mathfrak{R}_{\mathfrak{o}} + \mathfrak{R}_{\mathfrak{o}}\mathfrak{V}_{[i]} \subset \mathfrak{R}_{[i]}. \tag{9}$$

Finally, we also get $\mathfrak{R}_j\mathfrak{R}_k + \mathfrak{R}_k\mathfrak{R}_j \subset \mathfrak{R}_{[i]}$ for any $j \in [i]$ and $k \in \Delta$, being then

$$\Big(\bigoplus_{j\in\Delta}\mathfrak{R}_j\Big)\mathfrak{V}_{[i]} + \mathfrak{V}_{[i]}\Big(\bigoplus_{j\in\Delta}\mathfrak{R}_j\Big) \subset \mathfrak{R}_{[i]}. \tag{10}$$

From Equations (5)-(10) we conclude $\mathfrak{R}\mathfrak{R}_{[i]} + \mathfrak{R}_{[i]}\mathfrak{R} \subset \mathfrak{R}_{[i]}$. □

Corollary 2.1. *If $\mathfrak{R}$ is graded simple, then there exists a connection between any couple of elements in Δ, and $\mathfrak{R}_{\mathfrak{o}} = \sum\limits_{\{i,j\in\Delta: i\star j=\{\mathfrak{o}\}\}} \mathfrak{R}_i\mathfrak{R}_j$.*

Lemma 2.5. *For any $i,j \in \Delta$ such that $[i] \neq [j]$ we have that $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$.*

Proof. We can write

$$\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} \subset \mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{R}_{\mathfrak{o},[j]} + \mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{V}_{[j]} + \mathfrak{V}_{[i]}\mathfrak{R}_{\mathfrak{o},[j]} + \mathfrak{V}_{[i]}\mathfrak{V}_{[j]}. \tag{11}$$

By Proposition 2.2

$$\mathfrak{V}_{[i]}\mathfrak{V}_{[j]} \cap (\bigoplus_{k\in\Delta}\mathfrak{R}_k) \subset \mathfrak{V}_{[i]} \cap \mathfrak{V}_{[j]} = \{0\}. \tag{12}$$

Now observe that in case some $k \in [i]$ and $m \in [j]$ are such that $\{0\} \neq \mathfrak{R}_k\mathfrak{R}_m \subset \mathfrak{R}_{\mathfrak{o}}$ then $\{\overline{m}, \mathfrak{o}\}$ would be a connection from m to k being so $[i] = [j]$, a contradiction. From here

$$\mathfrak{V}_{[i]}\mathfrak{V}_{[j]} \cap \mathfrak{R}_{\mathfrak{o}} = \{0\}. \tag{13}$$

From Equations (12) and (13) and the grading of any $\mathfrak{V}_{[k]}$ we deduce

$$\mathfrak{V}_{[i]}\mathfrak{V}_{[j]} = \{0\}.$$

Hence

$$\begin{aligned}\mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{V}_{[j]} + \mathfrak{V}_{[i]}\mathfrak{R}_{\mathfrak{o},[j]} &\subset \sum_{a,b\in[i]}\mathfrak{R}_a(\mathfrak{R}_b\mathfrak{V}_{[j]}) + \sum_{c,d\in[j]}(\mathfrak{V}_{[i]}\mathfrak{R}_c)\mathfrak{R}_d \\ &\subset \sum_{a\in[i]}\mathfrak{R}_a(\mathfrak{V}_{[i]}\mathfrak{V}_{[j]}) + \sum_{d\in[j]}(\mathfrak{V}_{[i]}\mathfrak{V}_{[j]})\mathfrak{R}_d = \{0\}\end{aligned}$$

and

$$\mathfrak{R}_{\mathfrak{o},[i]}\mathfrak{R}_{\mathfrak{o},[j]} \subset \sum_{\substack{a,\,b\,\in\,[i]\\ c,\,d\,\in\,[j]}}(\mathfrak{R}_a\mathfrak{R}_b)(\mathfrak{R}_c\mathfrak{R}_d) \subset \sum_{(a,d)\in[i]\times[j]}\mathfrak{R}_a(\mathfrak{V}_{[i]}\mathfrak{V}_{[j]})\mathfrak{R}_d = \{0\}.$$

From the above, Equation (11) allows us to assert $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$. □

Theorem 2.1. *Let* $\mathfrak{U}$ *be a subgroup of* $\mathfrak{R}_{\mathfrak{o}}$ *such that* $\mathfrak{U} + \sum_{[i]\in\Delta/\sim} \mathfrak{R}_{\mathfrak{o},[i]} = \mathfrak{R}_{\mathfrak{o}}$. *Then*

$$\mathfrak{R} = \mathfrak{U} + \Big(\sum_{[i]\in\Delta/\sim} \mathfrak{R}_{[i]}\Big)$$

where $\{\mathfrak{R}_{[i]} : [i] \in \Delta/\sim\}$ *is a family of graded ideals satisfying* $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$ *when* $[i] \neq [j]$.

Proof. Since $\bigoplus_{i\in\Delta} \mathfrak{R}_i = \bigoplus_{[i]\in\Delta/\sim} \mathfrak{V}_{[i]}$, we have

$$\begin{aligned} \mathfrak{R} &= \Big(\mathfrak{U} + \sum_{[i]\in\Delta/\sim} \mathfrak{R}_{\mathfrak{o},[i]}\Big) \oplus \Big(\bigoplus_{[i]\in\Delta/\sim} \mathfrak{V}_{[i]}\Big) \\ &= \mathfrak{U} + \sum_{[i]\in\Delta/\sim} (\mathfrak{R}_{\mathfrak{o},[i]} \oplus \mathfrak{V}_{[i]}) \\ &= \mathfrak{U} + \Big(\sum_{[i]\in\Delta/\sim} \mathfrak{R}_{[i]}\Big). \end{aligned}$$

Proposition 2.2 and Lemma 2.5 complete the proof. □

If any element $\mathfrak{o}$ in the support of the grading is not distinguished, that is $\mathfrak{o} = \emptyset$, we have the next result as an immediate consequence of Theorem 2.1.

Corollary 2.2. *If* $\mathfrak{o} = \emptyset$ *then* $\mathfrak{R}$ *is the direct sum*

$$\mathfrak{R} = \bigoplus_{[i]\in\Sigma/\sim} \mathfrak{R}_{[i]}$$

where $\{\mathfrak{R}_{[i]} : [i] \in \Sigma/\sim\}$ *is a family of graded ideals satisfying* $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$ *when* $[i] \neq [j]$.

We recall that the *annihilator* of $\mathfrak{R}$ is the set

$$\mathrm{Ann}(\mathfrak{R}) = \{v \in \mathfrak{R} : v\mathfrak{R} + \mathfrak{R}v = \{0\}\},$$

and that, motivated by Corollary 2.1, we say that $\mathfrak{R}_{\mathfrak{o}}$ is *tight* whence

$$\mathfrak{R}_{\mathfrak{o}} = \sum_{\{i,j\in\Delta: i\star j=\{\mathfrak{o}\}\}} \mathfrak{R}_i\mathfrak{R}_j.$$

Corollary 2.3. *Suppose* $\mathrm{Ann}(\mathfrak{R}) = \{0\}$ *and* $\mathfrak{R}_0$ *is tight, then* $\mathfrak{R}$ *decomposes as the direct sum*

$$\mathfrak{R} = \bigoplus_{[i]\in\Delta/\sim} \mathfrak{R}_{[i]}$$

where $\{\mathfrak{R}_{[i]} : [i] \in \Delta/\sim\}$ *is a family of graded ideals satisfying* $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$ *when* $[i] \neq [j]$.

Proof. Since $\mathfrak{R}_{\mathfrak{o}}$ is tight we can take $\mathfrak{U} = 0$ in Theorem 2.1. From here, we just have to show the direct character of the sum. Given

$$x \in \mathfrak{R}_{[i]} \cap \sum_{[j]\neq[i]} \mathfrak{R}_{[j]},$$

taking into account $\mathfrak{R}_{[i]}\mathfrak{R}_{[j]} = \{0\}$ for $[i] \neq [j]$ we get $xR_{[i]} + R_{[i]}x = \{0\}$ and

$$x\Big(\sum_{[j]\neq[i]} R_{[j]}\Big) + \Big(\sum_{[j]\neq[i]} R_{[j]}\Big)x = \{0\}.$$

From the above $x\mathfrak{R} + \mathfrak{R}x = \{0\}$, that is, $x \in \mathrm{Ann}(\mathfrak{R}) = \{0\}$, as desired. □

3. The graded simple components

In this section we study when the components in the decompositions given in Theorem 2.1, Corollary 2.2 and Corollary 2.3 are graded simple. We begin by introducing the key notions of maximal length and Σ-multiplicativity in the setup of rings with an arbitrary grading, in a similar way to that for group-graded associative algebras, group-graded Lie algebras, group-graded Leibniz algebras and so on. For these notions and examples we refer to [1, 4, 5, 6].

From now on, for any $i \in \Sigma$ we will denote $\mathfrak{R}_{\overline{i}} := \{0\}$.

Definition 3.1. We say that $\mathfrak{R}$ is of *maximal length* if for any $i \in \Delta$, the only subgroups of $\mathfrak{R}_i$ are $\{0\}$ and itself.

Observe that any $\mathfrak{R}_i$ is an Abelian simple group and so isomorphic, as group, to $\mathbb{Z}_p$ with p prime.

Definition 3.2. We say that $\mathfrak{R}$ is Σ-multiplicative if for any $i \in \Sigma$ and $j, k \in \Sigma \,\dot{\cup}\, \overline{\Sigma}$ such that $i \in j \star k$ we have that $\mathfrak{R}_i \subset \mathfrak{R}(\mathfrak{R}_j + \mathfrak{R}_{\overline{j}})\mathfrak{R} \cap \mathfrak{R}(\mathfrak{R}_k + \mathfrak{R}_{\overline{k}})\mathfrak{R}$.

Example 3.1. Consider the $(I \times I \times J)$-graded ring

$$\mathfrak{S} = \bigoplus_{(i,j,k)\in I\times I\times J} \mathfrak{S}_{(i,j,k)}$$

where $\mathfrak{S}_{(i,j,k)} := \mathcal{R}E_{((i,k),(j,k))}$ of Example 1.1 and take $\mathcal{R} = \mathbb{Z}_p$ with p prime.

Observe that $\mathfrak{S}$ is of *maximal length*. We also have that $\mathfrak{S}$ is Σ-*multiplicative*. Indeed, if we take $(i,j,k), (i_1,j_1,k_1), (i_2,j_2,k_2) \in \Sigma$ such that

$$(i,j,k) \in (i_1,j_1,k_1) \star (i_2,j_2,k_2)$$

then necessarily $k = k_1 = k_2$, $i = i_1$, $j = j_2$ and $j_1 = i_2$. From here $(i,j,k) = (i_1,j_2,k_1)$ and since can write

$$E_{((i_1,k_1),(j_2,k_1))} = E_{((i_1,k_1),(i_1,k_1))}E_{((i_1,k_1),(j_1,k_1))}E_{((j_1,k_1),(j_2,k_1))}$$

and

$$E_{((i_1,k_1),(j_2,k_1))} = E_{((i_1,k_1),(j_1,k_1))}E_{((j_1,k_1),(j_2,k_1))}E_{((j_2,k_1),(j_2,k_1))}$$

we get $\mathfrak{S}_{(i,j,k)} \subset \mathfrak{S}\mathfrak{S}_{(i_1,j_1,k_1)}\mathfrak{S} \cap \mathfrak{S}\mathfrak{S}_{(i_2,j_2,k_2)}\mathfrak{S}$.

In case we take $(i,j,k), (i_2,j_2,k_2) \in \Sigma$ and $\overline{(i_1,j_1,k_1)} \in \overline{\Sigma}$ such that $(i,j,k) \in \overline{(i_1,j_1,k_1)} \star (i_2,j_2,k_2)$, we have that $k = k_1 = k_2$, and either $i = i_2$, $j = i_1$, $j_1 = j_2$ or $i = j_1$, $j = j_2$, $i_1 = i_2$, being then either $(i,j,k) = (i_2,i_1,k_1)$ or $(i,j,k) = (j_1,j_2,k_1)$. In the first possibility we can write

$$E_{((i_2,k_1),(i_1,k_1))} = E_{((i_2,k_1),(i_1,k_1))}E_{((i_1,k_1),(j_1,k_1))}E_{((j_1,k_1),(i_1,k_1))}$$

and

$$E_{((i_2,k_1),(i_1,k_1))} = E_{((i_2,k_1),(i_2,k_1))}E_{((i_2,k_1),(j_1,k_1))}E_{((j_1,k_1),(i_1,k_1))}$$

while in the second one

$$E_{((j_1,k_1),(j_2,k_1))} = E_{((j_1,k_1),(i_1,k_1))}E_{((i_1,k_1),(j_1,k_1))}E_{((j_1,k_1),(j_2,k_1))}$$

and

$$E_{((j_1,k_1),(j_2,k_1))} = E_{((j_1,k_1),(i_1,k_1))}E_{((i_1,k_1),(j_2,k_1))}E_{((j_2,k_1),(j_2,k_1))}.$$

From here $\mathfrak{S}_{(i,j,k)} \subset \mathfrak{S}\mathfrak{S}_{(i_1,j_1,k_1)}\mathfrak{S} \cap \mathfrak{S}\mathfrak{S}_{(i_2,j_2,k_2)}\mathfrak{S}$ in both possibilities.

Finally, if we take $(i,j,k),(i_1,j_1,k_1) \in \Sigma$ and $\overline{(i_2,j_2,k_2)} \in \overline{\Sigma}$ such that $(i,j,k) \in (i_1,j_1,k_1)\star\overline{(i_2,j_2,k_2)}$ we can argue as in the previous case to verify the Σ-multiplicativity of $\mathfrak{S}$.

Example 3.2. Take $I = \mathbb{N}$, $J = \{1,2,...,r\}$ a finite set, $\mathcal{R} = \mathbb{Z}_p$ and consider the graded ring $\mathfrak{S} = \bigoplus_{g\in G} \mathfrak{S}_g$ of Example 1.2 where $G = \mathbb{Q}^\times$, (the multiplicative rational group), and a family of r sequences of prime natural numbers $\{x_{n,t}\}_{n\in\mathbb{N}}$ where $t \in J$, such that $x_{n,t} \neq x_{m,s}$ when $(n,t) \neq (m,s)$ and define

$$\phi : \mathbb{N} \times J \to \mathbb{Q}^\times$$
$$(n,p) \mapsto x_{n,p}.$$

Taking into account Equation (1) it is easy to verify that for any $q \in \mathbb{Q}^\times$, $q \neq 1$, either $\mathfrak{S}_q = 0$ or $\mathfrak{S}_q = \mathbb{Z}_p E_{((n,t),(m,t))}$ for (unique) $n,m \in \mathbb{N}$ and $t \in J$ such that $x_{n,t}^{-1}x_{m,t} = q$.

We distinguish the element $\mathfrak{o} := 1 \in \mathbb{Q}^\times$ being then $\Delta = \Sigma \setminus \{1\}$ (see Example 2.1).

Observe that in case $\mathfrak{S}_q = \mathbb{Z}_p E_{((n,t),(m,t))}$ then $\mathfrak{S}_{q^{-1}} = \mathbb{Z}_p E_{((m,t),(n,t))}$ and that $\mathfrak{S}$ is of *maximal length*.

Since

$$\mathfrak{S}_1 = \bigoplus_{n\in\mathbb{N};\ t\in J} \mathbb{Z}_p E_{((n,t),(n,t))} \neq 0$$

and $E_{((n,t),(n,t))} = E_{((n,t),(m,t))}E_{((m,t),(n,t))}$ for any $m \in \mathbb{N}$ with $m \neq n$, we also get that $\mathfrak{S}_1 = \sum_{q\in\Delta} \mathfrak{S}_q\mathfrak{S}_{q^{-1}}$ and so $\mathfrak{S}_1$ is *tight*.

In order to verify that $\mathfrak{S}$ is Σ-multiplicative, take $i,q,p \in \Sigma$ such that $i \in q \star p$. By the above we can write $q = x_{n,t}^{-1}x_{m,t}$ and $p = x_{r,v}^{-1}x_{s,v}$. From here $t = v, m = r$ and so $i = x_{n,t}^{-1}x_{s,t}$. Thus

$$\mathfrak{S}_i = E_{((n,t),(n,t))}(\mathbb{Z}_p E_{((n,t),(m,t))})E_{((m,t),(s,t))} \subset \mathfrak{S}\mathfrak{S}_q\mathfrak{S}$$

and

$$\mathfrak{S}_i = E_{((n,t),(r,t))}(\mathbb{Z}_p E_{((r,t),(s,t))})E_{((s,t),(s,t))} \subset \mathfrak{S}\mathfrak{S}_p\mathfrak{S}.$$

If $i,p \in \Sigma$ and $q \in \overline{\Sigma}$ satisfy $i \in q \star p$ and write $\overline{q} = x_{n,t}^{-1}x_{m,t}$ and $p = x_{r,v}^{-1}x_{s,v}$ we have that necessarily either $(m,t) = (s,v)$ and $i = x_{r,t}^{-1}x_{n,t}$ or $(n,t) = (r,v)$ and $i = x_{m,t}^{-1}x_{s,t}$. From here we can write either

$$\mathfrak{S}_i = E_{((r,t),(n,t))}(\mathbb{Z}_p E_{((n,t),(m,t))})E_{((m,t),(n,t))} \subset \mathfrak{S}\mathfrak{S}_{\overline{q}}\mathfrak{S}$$

and

$$\mathfrak{S}_i = E_{((r,t),(r,t))}(\mathbb{Z}_p E_{((r,t),(s,t))})E_{((s,t),(n,t))} \subset \mathfrak{S}\mathfrak{S}_p\mathfrak{S}$$

or

$$\mathfrak{S}_i = E_{((m,t),(n,t))}(\mathbb{Z}_p E_{((n,t),(m,t))})E_{((m,t),(s,t))} \subset \mathfrak{S}\mathfrak{S}_{\overline{q}}\mathfrak{S}$$

and

$$\mathfrak{S}_i = E_{((m,t),(r,t))}(\mathbb{Z}_p E_{((r,t),(s,t))})E_{((s,t),(s,t))} \subset \mathfrak{S}\mathfrak{S}_p\mathfrak{S}.$$

Finally, since the case in which $i, q \in \Sigma$, $p \in \overline{\Sigma}$ and $i \in q \star p$ can be studied in a similar way we conclude that $\mathfrak{S}$ is Σ-*multiplicative*.

Theorem 3.1. *Let $\mathfrak{R}$ be a Σ-multiplicative arbitrarily graded ring of maximal length and with $\mathrm{Ann}(\mathfrak{R}) = \{0\}$. Then $\mathfrak{R}$ is graded simple if and only if it has all of the elements in Δ connected and $\mathfrak{R}_{\mathfrak{o}}$ is tight.*

Proof. The first implication is consequence of Corollary 2.1. To prove the converse, consider a nonzero graded ideal $I = \bigoplus_{i \in \Sigma_I} I_i$ of $\mathfrak{R}$ where $\Sigma_I = \{i \in \Delta : I \cap \mathfrak{R}_i \neq \{0\}\} \cup \{\mathfrak{o}\}$ and $I_j = I \cap \mathfrak{R}_j$ for any $j \in \Sigma_I$.

Denote by $\Delta_I = \Sigma_I \setminus \{\mathfrak{o}\}$. Since $\mathfrak{R}$ is of maximal length we have $\{0\} \neq I_i = I \cap \mathfrak{R}_i = \mathfrak{R}_i$ for any $i \in \Delta_I$ and so

$$I = I_{\mathfrak{o}} \oplus \Big(\bigoplus_{i \in \Delta_I} \mathfrak{R}_i \Big).$$

We assert that $\Delta_I \neq \emptyset$. Indeed, in case $I \subset \mathfrak{R}_{\mathfrak{o}}$, since $\mathrm{Ann}(\mathfrak{R}) = \{0\}$ we have that there exists some $k \in \Sigma$ satisfying $I\mathfrak{R}_k + \mathfrak{R}_k I \neq \{0\}$. From here $\{0\} \neq I\mathfrak{R}_k + \mathfrak{R}_k I \subset \mathfrak{R}_{\mathfrak{o}}$ and so necessarily $k = \mathfrak{o}$. The tight character of $\mathfrak{R}_{\mathfrak{o}}$ together with the associativity of the product give us now that there exists $i \in \Delta$ such that $\{0\} \neq I\mathfrak{R}_i + \mathfrak{R}_i I \subset \mathfrak{R}_{\mathfrak{o}}$, what contradicts the election of $\mathfrak{o}$. Hence $\Delta_I \neq \emptyset$.

From the above we can fix some $i_0 \in \Delta_I$ being then

$$\{0\} \neq \mathfrak{R}_{i_0} \subset I. \tag{14}$$

Let us now show by induction on n that if $\{a_1, \dots, a_n\}$ is any connection form i_0 to any $j \in \Delta$ then for any

$$b \in \phi(((\cdots \phi(\{\{a_1\}, a_2\}) \cdots), a_{n-1}), a_n) \cap \Delta$$

we have that

$$\mathfrak{R}_b \subset I.$$

In case $n = 2$, we get $b \in \phi(\{a_1\}, a_2)$ with $a_1 \in \{i_0, \overline{i_0}\}$. Hence

$$b \in i_0 \star a_2 \cup a_2 \star i_0 \cup \overline{i_0} \star a_2.$$

By Σ-multiplicativity and Equation (14) we obtain $\mathfrak{R}_b \subset I$.

Suppose now the assertion holds for any connection $\{b_1, \dots, b_n\}$ from i_0 to any $k \in \Delta$ and consider some arbitrary connection $\{a_1, \dots, a_{n+1}\}$ from i_0 to any $j \in \Delta$. We know that for any $c \in \mathfrak{A}$ where $\mathfrak{A} := \phi(((\cdots \phi(\{\{a_1\}, a_2\}) \cdots), a_{n-1}), a_n) \cap \Delta$, the subgroup

$$\mathfrak{R}_c \subset I. \tag{15}$$

Taking into account that any $b \in \phi(((\cdots \phi(\{\{a_1\}, a_2\}) \cdots), a_n), a_{n+1}) \cap \Delta$ means

$$b \in \phi(\mathfrak{A} \cup \overline{\mathfrak{A}}, a_{n+1}) \cap \Delta$$

we have $b \in c \star a_{n+1} \cup a_{n+1} \star c \cup \overline{c} \star a_{n+1}$ for some $c \in \mathfrak{A}$. From here, the Σ-multiplicativity of $\mathfrak{R}$ and Equation (15) allow us to get $\mathfrak{R}_b \subset I$ as desire.

Since given any $j \in \Delta$ we know i_0 is connected to j, we can assert by the above observation that $\mathfrak{R}_j \subset I$. We have shown

$$\bigoplus_{j \in \Delta} \mathfrak{R}_j \subset I.$$

From here, the tight character of $\mathfrak{R}_{\mathfrak{o}}$ gives us

$$\mathfrak{R}_{\mathfrak{o}} \subset I$$

and so $I = \mathfrak{R}$. □

Theorem 3.2. *Let $\mathfrak{R}$ be a Σ-multiplicative arbitrarily graded ring of maximal length, with $\mathrm{Ann}(\mathfrak{R}) = \{0\}$ and with $\mathfrak{R}_{\mathfrak{o}}$ tight. Then $\mathfrak{R}$ is the direct sum of the family of its minimal graded ideals, each one being a graded simple graded ring having all of the elements different to $\mathfrak{o}$ in its support connected.*

Proof. By Corollary 2.3 we have that

$$\mathfrak{R} = \bigoplus_{[i]\in\Delta/\sim} \mathfrak{R}_{[i]}. \tag{16}$$

We wish to apply Theorem 3.1 to any $\mathfrak{R}_{[i]}$ in the above decomposition. Clearly any $\mathfrak{R}_{[i]}$ is Σ-multiplicative and of maximal length as consequence of the Σ-multiplicativity and maximal length of $\mathfrak{R}$. The $\mathfrak{o}$-homogeneous component of $\mathfrak{R}_{[i]}$ is equals to $\mathfrak{R}_{\mathfrak{o},[i]}$ and so tight by construction. Also observe that in case $x \in \mathfrak{R}_{[i]}$ with $x\mathfrak{R}_{[i]} + \mathfrak{R}_{[i]}x = \{0\}$, then by Equation (16) and Lemma 2.5 we have $x \in \mathrm{Ann}(\mathfrak{R}) = \{0\}$. That is, $\mathrm{Ann}(\mathfrak{R}_{[i]}) = \{0\}$. Finally, since the set of element in the support of $\mathfrak{R}_{[i]}$ different to $\mathfrak{o}$ is $[i]$, and it is easy to verify that $[i]$ has all of its elements $[i]$-connected, (connected through elements in $[i] \dot\cup \overline{[i]}$), we can apply Theorem 3.1 to any $\mathfrak{R}_{[i]}$ so as to conclude $\mathfrak{R}_{[i]}$ is graded simple. □

Example 3.3. Let us consider the $(I \times I \times J)$-graded ring $\mathfrak{S}$ of Example 3.1. This is Σ-multiplicative and of maximal length. Clearly we also have $\mathrm{Ann}(\mathfrak{S}) = 0$.

Take now $(i, j, k), (n, m, r) \in \Sigma$ with $k = r$. Then the set

$$\{\overline{(i, j, k)}, (n, j, k), (i, m, k)\}$$

is a connection from (i, j, k) to (n, m, r).

However, if $k \neq r$, as for any $p, q \in I$ we have that $(i, j, k) \star (p, q, s) = \overline{(i, j, k)} \star (p, q, s) = \emptyset$ when $k \neq s$, and that $(i, j, k) \star (p, q, k) = \overline{(i, j, k)} \star (p, q, k) \in I \times I \times \{k\}$, we get that there is not any connection from (i, j, k) to (n, m, r). We have shown that the equivalence classes in $\Sigma/\sim$ are $[(i, j, k)] = I \times I \times \{k\}$ and by applying the results in this section we can assert that $\mathfrak{S}$ decomposes as the direct sum

$$\mathfrak{S} = \bigoplus_{k\in J} \mathfrak{S}_k$$

where any $\mathfrak{S}_k = \bigoplus_{(i,j)\in I\times I} \mathbb{Z}_p E_{((i,k),(j,k))}$ is a graded simple (minimal) graded ideal of $\mathfrak{S}$ having all of the elements of its support connected.

Example 3.4. Let us consider the graded ring $\mathfrak{S}$ of Example 3.2. This is Σ-multiplicative of maximal length and with $\mathfrak{S}_{\mathfrak{o}}$ tight. Also, it is easy to check that $\mathrm{Ann}(\mathfrak{S}) = 0$.

Take now $q, p \in \Delta$ with $p \neq q$ and recall that we can write $q = x_{n,t}^{-1}x_{m,t}$ with $n \neq m$ and $p = x_{r,v}^{-1}x_{s,v}$ with $r \neq s$. Suppose $v = t$. By fixing some $u, v \in \mathbb{N}$ such that $u \notin \{n, m, r\}$ and $v \notin \{m, r, s, u\}$ we get that the set

$$\{q, x_{u,t}^{-1}x_{n,t}, x_{m,t}^{-1}x_{v,t}, x_{r,t}^{-1}x_{u,t}, x_{v,t}^{-1}x_{s,t}\}$$

is a connection from q to p.

However, if $v \neq t$, and since $\Delta = \{x_{n,t}^{-1}x_{m,t} : n, m \in \mathbb{N} \text{ with } n \neq m \text{ and } t \in J\}$, there is not any connection from q to p. We have shown that the equivalence classes in $\Delta/\sim$ are

$[x_{n,t}^{-1}x_{m,t}] = \{x_{r,t}^{-1}x_{s,t} : r, s \in \mathbb{N}$ with $r \neq s\}$ and by applying the results in this section we can assert that, under the notation

$$\mathfrak{S}_t = (\sum_{n\in\mathbb{N}} \mathbb{Z}_p E_{((n,t),(n,t))}) \oplus (\bigoplus_{n,m\in\mathbb{N};n\neq m} \mathbb{Z}_p E_{((n,t),(m,t))})$$

for any $t \in J$, any $\mathfrak{S}_t$ is a graded simple, graded ring having all of the elements distinct to $\mathfrak{o} = 1$ of its support connected. Moreover, $\mathfrak{S}$ decomposes as the direct sum of these family of minimal graded ideals, namely:

$$\mathfrak{S} = \bigoplus_{t=1}^{r} \mathfrak{S}_t.$$

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