Manuscript

MONOMIAL IDEALS INDUCED BY PERMUTATIONS AVOIDING PATTERNS

AJAY KUMAR AND CHANCHAL KUMAR

Abstract. Let S (or T) be the set of permutations of $[n] = \{1, \ldots, n\}$ avoiding 123 & 132-patterns (or avoiding 123, 132 & 213-patterns). The monomial ideals $I_S = \left\langle \mathbf{x}^\sigma = \prod_{i=1}^n x_i^{\sigma(i)} : \sigma \in S \right\rangle$ and $I_T = \langle \mathbf{x}^\sigma : \sigma \in T \rangle$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k have many interesting properties. The Alexander dual $I_S^{[\mathbf{n}]}$ of I_S w.r.t. $\mathbf{n} = (n, \ldots, n)$ has the minimal cellular resolution supported on the order complex $\mathbf{\Delta}(\Sigma_n)$ of a poset Σ_n. The Alexander dual $I_T^{[\mathbf{n}]}$ also has the minimal cellular resolution supported on the order complex $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ of a poset $\widetilde{\Sigma}_n$. The number of standard monomials of the Artinian quotient $\frac{R}{I_S^{[\mathbf{n}]}}$ is given by the number of *irreducible* (or *indecomposable*) permutations of $[n+1]$, while the number of standard monomials of the Artinian quotient $\frac{R}{I_T^{[\mathbf{n}]}}$ is given by the number of permutations of $[n+1]$ having no substring $\{l, l+1\}$.

Key words: Permutations avoiding patterns, cellular resolutions, standard monomials, parking functions.

1. Introduction

Many classes of monomial ideals I in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k have the property that the number of standard monomials in the Artinian quotient $\frac{R}{I}$ is given in terms of determinant of a square matrix. For many combinatorially defined monomial ideals I, the standard monomials in $\frac{R}{I}$ correspond to suitable combinatorial objects. For an oriented graph (digraph) G on the vertex set $\{0, 1, \ldots, n\}$ rooted at 0, Postnikov and Shapiro [9] associated a monomial ideal $\mathcal{M}_G$ in R such that the Artinian quotient $\frac{R}{\mathcal{M}_G}$ has a standard monomial basis corresponding to the G-parking functions and the number of G-parking functions equals the number of (oriented) spanning trees of G, i.e., $\dim_k(\frac{R}{\mathcal{M}_G}) = \det(L_G)$, where L_G is the *truncated Laplace matrix* of G. More precisely, if $A = [a_{ij}]_{0 \le i,j \le n}$ is the adjacency matrix of the oriented graph G, then the monomial ideal $\mathcal{M}_G$ is given by

$$\mathcal{M}_G = \left\langle \prod_{i \in I} x_i^{d_I(i)} : I \in \Sigma \right\rangle,$$

2010 *Mathematics Subject Classification.* 05E40, 13D02.

where $d_I(i) = \sum_{j\in\{0,1,\ldots,n\}-I} a_{ij}$ is the number of (oriented) edges from the vertex i to a vertex outside of the subset I and Σ is the poset of all non-empty subsets of $[n]$ (ordered by inclusion). Also, $L_G = [l_{ij}]_{1\leq i,j\leq n}$ is given by $l_{ij} = \begin{cases} d_{\{i\}}(i) & \text{if } i = j, \\ -a_{ij} & \text{if } i \neq j. \end{cases}$ The adjacency matrix of a (non-oriented) graph is symmetric and therefore a graph can be identified with a unique oriented graph having the same (symmetric) adjacency matrix. Under this identification, oriented spanning trees correspond to usual spanning trees of the graph. Therefore, notion of G-parking functions also make sense for a graph G. An oriented graph G with adjacency matrix $A = [a_{ij}]$ is called *saturated* if $a_{ij} \geq 1$ for $i \neq j$. For a saturated graph G, the monomial ideal $\mathcal{M}_G$ is an *order monomial ideal* (Definition 2.3) and its minimal resolution is the cellular resolution supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex Δ_{n-1} (see [9](Corollary 6.9)). In case, G is a complete graph K_{n+1}, the monomial ideal

$$\mathcal{M}_{K_{n+1}} = \left\langle \left(\prod_{i\in I} x_i\right)^{n-|I|+1} : I \in \Sigma \right\rangle$$

is called a *tree ideal*. Further, we see that a K_{n+1}-parking function is an (*ordinary*) *parking function* of length n, which is defined as a sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ with $0 \leq p_i < n$ such that the non-decreasing rearrangement $q_1 \leq \ldots \leq q_n$ of $\mathbf{p}$ satisfies $q_i < i$ (or equivalently, $|\{j \in [n] : p_j < i\}| \geq i, \ \forall i \in [n]$).

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 1$, the monomial ideal $I_\lambda = \left\langle \left(\prod_{i\in A} x_i\right)^{\lambda_{|A|}} : \emptyset \neq A \subseteq [n] \right\rangle$ in R has an Artinian quotient $\frac{R}{I_\lambda}$ having a standard monomial basis corresponding to λ-parking functions. A sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a *λ-parking function of length n*, if its non-decreasing rearrangement $q_1 \leq q_2 \leq \ldots \leq q_n$ satisfies $q_i < \lambda_{n-i+1}, \ \forall i$. The (ordinary) parking functions of length n correspond to $\lambda = (n, n-1, \ldots, 1)$. Also, there is a Steck determinant formula for counting the number of λ-parking functions. Further, if $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, then the minimal cellular resolution of I_λ is supported on $\mathbf{Bd}(\Delta_{n-1})$ [9]. For more on λ-parking functions, we refer to [8, 12]. The multigraded Betti numbers of I_λ for any λ are given in [4].

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. Let S be the subset of permutations $\sigma \in \mathfrak{S}_n$ avoiding 123 & 132-patterns and let T be the subset of permutations $\sigma \in \mathfrak{S}_n$ avoiding 123, 132 & 213-patterns. Then, it is shown in [10] that $|S| = 2^{n-1}$ and $|T| = \mathrm{F}_{n+1}$, where F_n is the nth Fibonacci number (i.e., $\mathrm{F}_0 = 0, \mathrm{F}_1 = 1$ and $\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}; \ n \geq 2$).

Now consider the monomial ideals $I_S = \langle \mathbf{x}^\sigma : \sigma \in S \rangle$ and $I_T = \langle \mathbf{x}^\sigma : \sigma \in T \rangle$ in $R = k[x_1, \ldots, x_n]$ induced by subsets S and T, respectively. The minimal generators of the Alexander dual $I_S^{[\mathbf{n}]}$ of I_S with respect to $\mathbf{n} = (n, \dots, n)$ are given by (Lemma 2.1)

$$I_S^{[\mathbf{n}]} = \left\langle x_l^{l+1}, \quad \left(\prod_{j=m}^{n} x_j\right)^m : 1 \leq l \leq n-1, \quad 1 \leq m \leq n \right\rangle.$$

Similarly, the minimal generators of the Alexander dual $I_T^{[\mathbf{n}]}$ are given by

$$I_T^{[\mathbf{n}]} = \left\langle x_l^{l+1}, \quad \left(\prod_{j\in[m,m+1]} x_j\right)^m : 1 \leq l \leq n-1, \quad 1 \leq m \leq n \right\rangle,$$

where $[m, m+1] = \{m, m+1\}$ for $1 \leq m \leq n-1$ and $[n, n+1]$ stands for $\{n\}$. The monomial ideal $I_{\mathfrak{S}_n} = \langle \mathbf{x}^\sigma : \sigma \in \mathfrak{S}_n \rangle$ is called a *permutohedron ideal* and the Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]} = \mathcal{M}_{K_{n+1}} = I_\lambda$ for $\lambda = (n, n-1, \dots, 1)$.

Let $\Sigma_n = \{\{l\} : 1 \leq l \leq n-1\} \bigcup \{[m, n] : 1 \leq m \leq n\}$, where $[a, b] = \{x \in \mathbb{N} : a \leq x \leq b\}$ denotes an integer interval for $1 \leq a \leq b \leq n$. We define a partial ordering $\preceq$ on Σ_n as follows: for $l, l' \in [n-1]$ and $m, m' \in [n]$, $\{l\} \preceq \{l'\} \preceq [m, n]$ if $m \leq l' \leq l$ and $[m, n] \preceq [m', n] \preceq \{l\}$ if $l+1 < m' \leq m$. Consider the order complex $\mathbf{\Delta}(\Sigma_n)$ of the poset $(\Sigma_n, \preceq)$. An r-dimensional face of $\mathbf{\Delta}(\Sigma_n)$ is a (strict) chain $C_1 \prec C_2 \prec \ldots \prec C_{r+1}$ of length r in Σ. Let $f_r(\mathbf{\Delta}(\Sigma_n))$ be the number of r-dimensional faces of $\mathbf{\Delta}(\Sigma_n)$. Then, we prove that (Theorem 2.7)

$$f_r(\mathbf{\Delta}(\Sigma_n)) = \sum_{s=0}^{r+1} \binom{n-1}{s} \binom{n-s}{r+1-s}, \quad (0 \leq r \leq n-1).$$

Let $\widetilde{\Sigma}_n = \{\{l\} : 1 \leq l \leq n-1\} \bigcup \{[m, m+1] : 1 \leq m \leq n\}$, where $[m, m+1] = \{m, m+1\}$ for $1 \leq m \leq n-1$ and $[n, n+1] = \{n\}$. We define a partial ordering $\preceq'$ on $\widetilde{\Sigma}_n$ as follows: for $l, l' \in [n-1]$ and $m, m' \in [n]$, $[m, m+1] \prec' \{l\} \prec' \{l'\}$ if $l'+1 < l < m-1$ and $\{l\} \preceq' [m, m+1] \preceq' [m', m'+1]$ if $m' \leq m \leq l$. The order complex $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ of the poset $(\widetilde{\Sigma}_n, \preceq')$ is a simplicial complex of dimension $n-1$. We prove that (Theorem 2.7) the number $f_r(\mathbf{\Delta}(\widetilde{\Sigma}_n))$ of r-dimensional faces of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ is given by

$$f_r(\mathbf{\Delta}(\widetilde{\Sigma}_n)) = \sum_{s=0}^{r+1} \binom{n-s}{s} \binom{n-s}{r+1-s}, \quad (0 \leq r \leq n-1).$$

We label the vertices $\{l\}$ or $[m, n]$ of $\mathbf{\Delta}(\Sigma_n)$ by monomials x_l^{l+1} or $\left(\prod_{j\in[m,n]} x_j\right)^m$, respectively. Similarly, the vertices $\{l\}$ and $[m, m+1]$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ can be naturally labelled with monomials x_l^{l+1} or $\left(\prod_{j\in[m,m+1]} x_j\right)^m$, respectively. Now labelling the faces F by the LCM of monomial labels on the vertices of F, we see that the order complexes $\mathbf{\Delta}(\Sigma_n)$ and $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ are both labelled simplicial complexes. Both the ideals $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ are order monomial ideals (Proposition 2.5). In view of Theorem 2.4, the free complex associated to the labelled simplicial complexes $\mathbf{\Delta}(\Sigma_n)$ and $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ give the minimal cellular resolution of $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$, respectively. Thus Betti numbers of $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ are given by

$$\beta_i\left(I_S^{[\mathbf{n}]}\right) = f_i(\mathbf{\Delta}(\Sigma_n)) \quad \text{and} \quad \beta_i\left(I_T^{[\mathbf{n}]}\right) = f_i(\mathbf{\Delta}(\widetilde{\Sigma}_n)),$$

for $0 \leq i \leq n-1$. For more on cellular resolutions, we refer to [1, 2, 6].

We show that the standard monomial basis of $\frac{R}{I_S^{[\mathbf{n}]}}$ is given by $\mathbf{x}^{\mathbf{p}}$, where $\mathbf{p} = (p_1, \ldots, p_n)$ is a parking function of length n satisfying $p_i \leq i,\ \forall i$. Such parking functions may be called *Catalan parking functions*. Let Λ_n be the set of all parking functions of length n. Then $|\Lambda_n| = (n+1)^{n-1}$. Let Λ_n^{Cat} be the set of Catalan parking functions of length n. We show that (Theorem 3.4)

$$|\Lambda_n^{\mathrm{Cat}}| = \dim_k\left(\frac{R}{I_S^{[\mathbf{n}]}}\right) = (-1)^n \det\left([m_{ij}]_{(n+1)\times(n+1)}\right),$$

where $m_{ij} = \begin{cases} (j-i+1)! & \text{if } i \leq j+1, \\ 0 & \text{if } i > j+1. \end{cases}$ Further, it is observed that the number of Catalan parking functions of length n equals the number of *irreducible* (or *indecomposable*) permutations of $[n+1]$.

Since $I_T \subseteq I_S$, we have $I_S^{[\mathbf{n}]} \subseteq I_T^{[\mathbf{n}]}$. Thus a standard monomial $\mathbf{x}^{\mathbf{p}}$ of $\frac{R}{I_T^{[\mathbf{n}]}}$ is also a standard monomial of $\frac{R}{I_S^{[\mathbf{n}]}}$. We observe that $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_T^{[\mathbf{n}]}}$ if and only if $\mathbf{p} = (p_1, \ldots, p_n)$ is a Catalan parking function of length n such that for $1 \leq i \leq n-1$, if $p_i = i$, then $p_{i+1} < i$. A Catalan parking function $\mathbf{p} = (p_1, \ldots, p_n)$ of length n such that either $p_i < i$ or $p_{i+1} < i$ for every $i \in [n-1]$ is called a *restricted Catalan parking function* of length n. Let $\widetilde{\Lambda}_n^{\mathrm{Cat}}$ be the set of all restricted Catalan parking functions of length n. We show that (Theorem 4.5)

$$|\widetilde{\Lambda}_n^{\mathrm{Cat}}| = \dim_k\left(\frac{R}{I_T^{[\mathbf{n}]}}\right) = \det\left([\widetilde{m}_{ij}]_{n\times n}\right),$$

where $\widetilde{m}_{ij} = \begin{cases} j & \text{if } i = j \text{ or } i = j+1, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$ The number of restricted Catalan parking functions of length n equals the number of permutations of $[n+1]$ having no substring $\{l, l+1\}$.

In the last section, we have discussed some generalizations.

2. Betti numbers of $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$

Let $t \leq n$ be positive integers and τ be a fixed permutation of $[t]$ called a *pattern*. A permutation $\sigma \in \mathfrak{S}_n$ is said to *avoid the pattern* τ if there does not exist integers $1 \leq j_1 < \ldots < j_t \leq n$ such that for all $1 \leq a < b \leq t$, we have $\tau(a) < \tau(b)$ if and only if $\sigma(j_a) < \sigma(j_b)$. Let S and T be the subsets of $\mathfrak{S}_n$ as defined in the introduction. In this section, we study Alexander duals $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ of the the monomial ideals I_S and I_T. The Alexander dual $I_S^{[\mathbf{n}]}$ of I_S with respect to $\mathbf{n} = (n, \ldots, n)$ is a monomial ideal in R and a vector $\mathbf{b} = (b_1, \ldots, b_n) \leq \mathbf{n}$ (i.e., $b_i \leq n \ \forall i$) is maximal with $\mathbf{x}^{\mathbf{b}} \notin I_S$ if and only if $\mathbf{x}^{\mathbf{n}-\mathbf{b}} = \prod_{j=1}^{n} x_j^{n-b_j}$ is a minimal generator of $I_S^{[\mathbf{n}]}$ [6, 7].

Lemma 2.1. *The minimal generators of $I_S^{[\mathbf{n}]}$ are given by*

$$I_S^{[\mathbf{n}]} = \left\langle x_l^{l+1}, \left(\prod_{j \in [m,n]} x_j \right)^m : 1 \leq l \leq n-1 \ \& \ 1 \leq m \leq n \right\rangle.$$

Proof. For any $l \in [n-1]$, let $\mathbf{b}_l = (n, \ldots, n-l-1, \ldots, n)$ (i.e. $n-l-1$ at l^{th} place and elsewhere n). We claim that $\mathbf{x}^{\mathbf{b}_l} \notin I_S$. If not, then there is a $\sigma \in S$ such that $\mathbf{x}^{\sigma}$ divides $\mathbf{x}^{\mathbf{b}_l}$. Thus $1 \leq \sigma(l) \leq n-l-1$. This implies that $l \leq n-2$. Also, $|[l+1, n]| = n-l$ and $|\{a \in [n] : a < \sigma(l)\}| \leq n-l-2$ ensure that there exist $i, j \in [l+1, n]$ such that $\sigma(l) < \sigma(i) < \sigma(j)$. But, then σ contains either 123 or 132-pattern, a contradiction. Further, for any vector $\mathbf{b}_l'$ with $\mathbf{b}_l < \mathbf{b}_l' \leq \mathbf{n}$, $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}_l'}$ for $\sigma' = (n-1, n-2, \ldots, 1, n) \in S$. This gives the minimal generators x_l^{l+1} for all $l \in [n-1]$. For $[m, n]$, we take $\mathbf{b}_{[m,n]} = (n, \ldots, n, n-m, \ldots, n-m)$ (i.e. the last $n-m+1$ coordinates are $n-m$, elsewhere n). Again, $\mathbf{x}^{\mathbf{b}_{[m,n]}} \notin I_S$, otherwise there is a $\sigma \in S$ such that $\mathbf{x}^{\sigma}$ divides $\mathbf{x}^{\mathbf{b}_{[m,n]}}$. Thus $\sigma(i) \leq n-m \ \forall i \in [m, n]$. Since $|[m, n]| = n-m+1$ and $|[1, n-m]| = n-m$, by the pigeon-hole principle, no such permutation σ exist. Also, if $\mathbf{b}_{[m,n]} < \mathbf{b}'_{[m,n]} \leq \mathbf{n}$, then we have $\mathbf{x}^{\mathbf{b}'_{[m,n]}} \in I_S$. This gives the minimal generator $\left(\prod_{j \in [m,n]} x_j \right)^m$. □

As in the Lemma 2.1, we compute the minimal generators of $I_T^{[\mathbf{n}]}$.

Lemma 2.2. *The minimal generators of $I_T^{[\mathbf{n}]}$ are given by*

$$I_T^{[\mathbf{n}]} = \left\langle x_l^{l+1}, \left(\prod_{j \in [m,m+1]} x_j \right)^m : 1 \leq l \leq n-1 \ \& \ 1 \leq m \leq n \right\rangle,$$

where $[m, m+1] = \{m, m+1\}$ for $m \in [n-1]$ and $[n, n+1] = \{n\}$.

Proof. Proceeding as in the last lemma, we see that x_l^{l+1} is a minimal generator of $I_T^{[\mathbf{n}]}$ for all $l \in [n-1]$. For $m \in [n-1]$, we take $\mathbf{b}_{[m,m+1]} = (n,\ldots,n,n-m,n-m,\ldots,n)$ (i.e. mth and $(m+1)$th coordinates are $n-m$, elsewhere n). Also, $\mathbf{b}_{[n,n+1]} = (n,\ldots,n,0)$ (i.e. nth coordinate is 0 and elsewhere n). We claim that $\mathbf{x}^{\mathbf{b}_{[m,m+1]}} \notin I_T$. Otherwise, there is a $\sigma \in T$ such that $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}_{[m,m+1]}}$. For $m = n$, we have $\sigma(n) \leq 0$, and for $m = n-1$, we must have $\sigma(n-1) \leq 1$ and $\sigma(n) \leq 1$. Such a permutation σ is not possible. Also, for $m = 1$, we have $\sigma(1) \leq n-1$ and $\sigma(2) \leq n-1$. But, for $\sigma \in T$ it can be checked that either $\sigma(1) = n$ or $\sigma(2) = n$. Thus, we assume that $2 \leq m \leq n-2$. Then $\sigma(m) \leq n-m$ and $\sigma(m+1) \leq n-m$. Since $|[m,n]| = n-m+1$ and $|[1,n-m]| = n-m$, by the pigeon-hole principle, there exists $l \in [m+2,n]$ such that $n-m < \sigma(l)$. If $\sigma(m) < \sigma(m+1)$, then permutation σ has a 123-pattern and if $\sigma(m) > \sigma(m+1)$, then it has a 213-pattern. Since $\sigma \in T$, this is not possible. Also, it is easy to verify that $\mathbf{b}_{[m,m+1]} \leq \mathbf{n}$ is a maximal vector such that $\mathbf{x}^{\mathbf{b}_{[m,m+1]}} \notin I_T$. This gives the minimal generator $\left(\prod_{j\in[m,m+1]} x_j\right)^m$. □

We proceed to show that ideals $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ are both order monomial ideals. Order monomial ideals are introduced and studied in [9].

Definition 2.3. Let P be a finite poset. Let $\{\omega_u : u \in P\}$ be a collection of monomials in $k[x_1,\ldots,x_n]$. The ideal $I = \langle \omega_u : u \in P\rangle$ is called an *order monomial ideal* if for any pair $u, v \in P$, there is an upper bound $w \in P$ of u and v such that ω_w divides the least common multiple $\mathrm{LCM}(\omega_u, \omega_v)$ of ω_u and ω_v.

Now we state a result of Postnikov and Shapiro [9](Theorem 6.1) in terms of cellular resolution. Let Δ be a labelled simplicial (or polyhedral) cell complex and $\mathbb{F}_*(\Delta)$ be the free complex associated to Δ (see [6]).

Theorem 2.4 (Postnikov-Shapiro)**.** *Let $I = \langle \omega_u : u \in P\rangle$ be an order monomial ideal. Then the free complex $\mathbb{F}_*(\boldsymbol{\Delta}(P))$ supported on the order complex $\boldsymbol{\Delta}(P)$ is a cellular resolution of I. Further, the cellular resolution $\mathbb{F}_*(\boldsymbol{\Delta}(P))$ is minimal if the monomial label on any face of $\boldsymbol{\Delta}(P)$ is different from the monomial labels on its proper subfaces.*

Let $(\Sigma_n, \preceq)$ and $(\widetilde{\Sigma}_n, \preceq')$ be the posets defined in the introduction. Let $\boldsymbol{\Delta}(\Sigma_n)$ and $\boldsymbol{\Delta}(\widetilde{\Sigma}_n)$ be associated order (simplicial) complexes. If F is an $i-1$-dimensional face of $\boldsymbol{\Delta}(\Sigma_n)$ corresponding to a (strict) chain $C_1 \prec \ldots \prec C_i$ of length $i-1$ in Σ_n, then the monomial label $\mathbf{x}^{\nu(F)}$ on F is given by

$$\mathbf{x}^{\nu(F)} = \prod_{q=1}^{i}\left(\prod_{j\in C_q - C_{q-1}} x_j^{\nu_{j,C_q}}\right),$$

where $C_0 = \emptyset$ and

$$\nu_{j,C_q} = \begin{cases} l+1 & \text{if } C_q = \{l\}, \\ m & \text{if } C_q = [m,n]. \end{cases} \tag{2.1}$$

Similarly, if $\widetilde{F}$ is an $i-1$-dimensional face of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ corresponding to a (strict) chain $\tilde{C}_1 \prec' \ldots \prec' \tilde{C}_i$ of length $i-1$ in $\widetilde{\Sigma}_n$, then the monomial label $\mathbf{x}^{\mu(\widetilde{F})}$ on $\widetilde{F}$ is given by

$$\mathbf{x}^{\mu(\widetilde{F})} = \prod_{q=1}^{i} \left(\prod_{j \in \tilde{C}_q - \tilde{C}_{q-1}} x_j^{\mu_{j,\tilde{C}_q}} \right),$$

where $\tilde{C}_0 = \emptyset$ and

$$\mu_{j,\tilde{C}_q} = \begin{cases} l+1 & \text{if } \tilde{C}_q = \{l\}, \\ m & \text{if } \tilde{C}_q = [m,m+1]. \end{cases} \tag{2.2}$$

Proposition 2.5. *The ideals $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ are both order monomial ideals in R.*

Proof. It is clear that $I_S^{[\mathbf{n}]} = \langle \omega_u : u \in \Sigma_n \rangle$ and $I_T^{[\mathbf{n}]} = \langle \omega_{\tilde{u}} : \tilde{u} \in \widetilde{\Sigma}_n \rangle$, where

$$\omega_{\{l\}} = x_l^{l+1}, \quad \omega_{[m,n]} = \left(\prod_{j \in [m,n]} x_j \right)^m \quad \text{and} \quad \omega_{[m,m+1]} = \left(\prod_{j \in [m,m+1]} x_j \right)^m.$$

Let u, v be a pair of elements of Σ_n. If u and v are comparable, then an upper bound w of u and v is given by $w = \begin{cases} v & \text{if } u \preceq v, \\ u & \text{if } v \preceq u. \end{cases}$ Clearly, ω_w divides $\mathrm{LCM}(\omega_u, \omega_v)$. If u and v are non-comparable, then $\{u, v\} = \{\{i\}, [i+1, n]\}$ for some $i < n$. Clearly, $w = [i, n]$ is an upper bound of u and v such that ω_w divides $\mathrm{LCM}(\omega_u, \omega_v)$. Similarly, if we take a pair of non-comparable elements $\tilde{u}, \tilde{v}$ in $\widetilde{\Sigma}_n$, then $\{\tilde{u}, \tilde{v}\} = \{\{i\}, \{i+1\}\}$ or $\{\{i\}, [i+1, i+2]\}$ for some $i < n$. In either of the cases, we take an upper bound $\tilde{w} = [i, i+1]$ of $\tilde{u}$ and $\tilde{v}$ and see that $\omega_{\tilde{w}}$ divides $\mathrm{LCM}(\omega_{\tilde{u}}, \omega_{\tilde{v}})$. This completes the proof. □

Example 2.6. For $n = 1$ or 2, we have $S = T = \mathfrak{S}_n$ and hence $I_S^{[\mathbf{n}]} = I_T^{[\mathbf{n}]} = I_{\mathfrak{S}_n}^{[\mathbf{n}]}$. Thus we consider these ideals for $n = 3$. We have $I_S^{[\mathbf{3}]} = \langle x_1^2, x_2^3, x_3^3, x_1x_2x_3, x_2^2x_3^2 \rangle$ and $I_T^{[\mathbf{3}]} = \langle x_1^2, x_2^3, x_3^3, x_1x_2, x_2^2x_3^2 \rangle$, while $I_{\mathfrak{S}_3}^{[\mathbf{3}]}$ is a tree ideal. The Hasse diagrams of posets $(\Sigma_3, \preceq)$, $(\widetilde{\Sigma}_3, \preceq')$ and $(\Sigma, \subseteq)$ are given in the Figure-1 and their order complexes with monomial vertex labels are indicated in Figure-2. In Figure-1, vertices are subsets of $\{1, 2, 3\}$, which are represented by an array of elements.

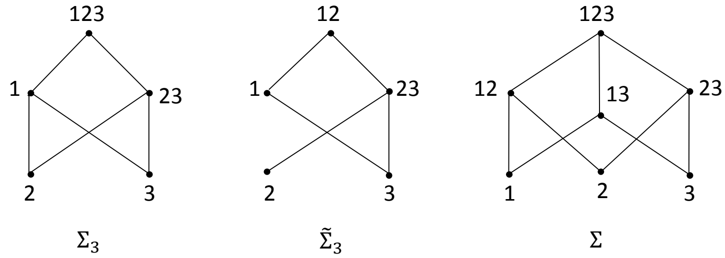

FIGURE 1. Hasse diagrams of $\Sigma_3, \widetilde{\Sigma}_3$ and Σ

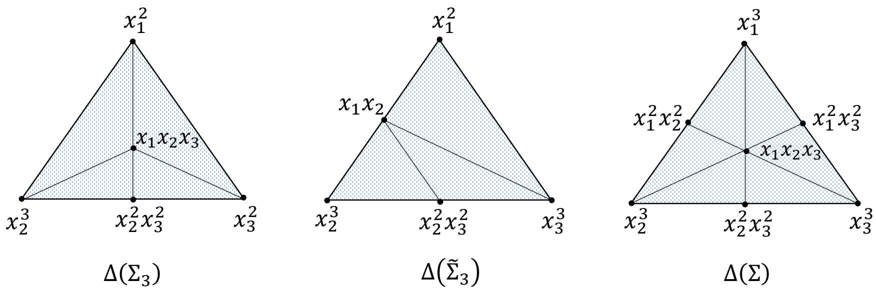

FIGURE 2. Order complexes with monomial labels on vertices

In view of Theorem 2.4, the cellular resolution supported on $\mathbf{\Delta}(\Sigma_n)$ (or $\mathbf{\Delta}(\widetilde{\Sigma}_n)$) gives the minimal resolution of $I_S^{[\mathbf{n}]}$ (respectively, $I_T^{[\mathbf{n}]}$). Thus i^{th} Betti numbers $\beta_i(I_S^{[\mathbf{n}]}) = f_i(\mathbf{\Delta}(\Sigma_n))$ and $\beta_i(I_T^{[\mathbf{n}]}) = f_i(\mathbf{\Delta}(\widetilde{\Sigma}_n))$, where $f_i(\Delta)$ denotes the number of i-dimensional faces of a simplicial complex Δ.

Theorem 2.7. *For* $0 \leq r \leq n-1$,

(a) $\beta_r(I_S^{[\mathbf{n}]}) = f_r(\mathbf{\Delta}(\Sigma_n)) = \sum_{s=0}^{r+1} \binom{n-1}{s}\binom{n-s}{r+1-s}$.

(b) $\beta_r(I_T^{[\mathbf{n}]}) = f_r(\mathbf{\Delta}(\widetilde{\Sigma}_n)) = \sum_{s=0}^{r+1} \binom{n-s}{s}\binom{n-s}{r+1-s}$.

Proof. (a) There are $n-1$ singletons $\{l\}$ and n integer intervals $[m,n]$ in the poset Σ_n. An r-dimensional face of $\mathbf{\Delta}(\Sigma_n)$ is a (strict) chain

$$C_1 \prec C_2 \prec \ldots \prec C_{r+1} \tag{2.3}$$

of length r in Σ_n. Suppose exactly s members in the chain (2.3) are singletons. Any two singletons (or any two integer intervals) in Σ_n are comparable but a singleton $\{l\}$ is comparable to an integer interval $[m,n]$ if and only if $m \neq l+1$. Also, for s singleton members in (2.3), exactly s integer intervals cannot occur in the chain. Now s singleton members in the chain (2.3) can be chosen in $\binom{n-1}{s}$ ways, and for each such choice, remaining $r+1-s$ integer intervals in the chain can be chosen in $\binom{n-s}{r+1-s}$ ways. Thus total number of chains in Σ of length r having exactly s singleton members is $\binom{n-1}{s}\binom{n-s}{r+1-s}$. As s varies from 0 to $r+1$, we get part(a).

(b) An r-dimensional face of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ is a (strict) chain

$$\tilde{C}_1 \prec' \tilde{C}_2 \prec' \ldots \prec' \tilde{C}_{r+1} \tag{2.4}$$

of length r in $\widetilde{\Sigma}_n$. Suppose exactly s members in the chain (2.4) are singletons. Any two non-consecutive singletons in $\widetilde{\Sigma}_n$ are comparable and s singletons in the chain form a s-subset of $[n-1]$ having no consecutive elements. Number of such s-subsets is precisely $\binom{n-s}{s}$. Also, for s singleton members in the chain (2.4), exactly s integer intervals cannot occur in the chain. Now proceeding as in the part(a), we obtain part(b). □

Miller, Sturmfels and Yanagawa [5] defined *generic* and *strongly generic* monomial ideals. For more on generic ideals, we refer to [6]. We end this section with the following remarks.

Remarks 2.8. (1) The tree ideal $\mathcal{M}_{K_{n+1}} = I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is generic and therefore, minimal resolution of the tree ideal is supported on its Scarf complex (see [6](Theorem 6.13)). The Scarf complex of the tree ideal $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is isomorphic to the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex Δ_{n-1}.
(2) The ideals $I_S^{[\mathbf{n}]}$ and $I_T^{[\mathbf{n}]}$ are in fact strongly generic. Thus the Scarf complex of $I_S^{[\mathbf{n}]}$ (or $I_T^{[\mathbf{n}]}$) is isomorphic to the order complex $\mathbf{\Delta}(\Sigma_n)$ (respectively, $\mathbf{\Delta}(\widetilde{\Sigma}_n)$) (see [9](Lemma 6.5)).

3. Catalan parking functions

The standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ are of the form $\mathbf{x}^{\mathbf{p}}$, where $\mathbf{p}$ is an (ordinary) parking function of length n. Since $I_S \subseteq I_{\mathfrak{S}_n}$, we have $I_{\mathfrak{S}_n}^{[\mathbf{n}]} \subseteq I_S^{[\mathbf{n}]}$. Thus every standard monomial of $\frac{R}{I_S^{[\mathbf{n}]}}$ is also a standard monomial of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$. We now characterize the standard monomials of $\frac{R}{I_S^{[\mathbf{n}]}}$.

Lemma 3.1. *For a parking function $\mathbf{p} = (p_1, \ldots, p_n)$ of length n, $\mathbf{x}^{\mathbf{p}} \notin I_S^{[\mathbf{n}]}$ if and only if $p_i \leq i$, $\forall i \in [n]$.*

Proof. We see that $\mathbf{x}^{\mathbf{p}} \in I_S^{[\mathbf{n}]}$ if and only if either $p_l \geq l+1$ for some $l \in [n-1]$ or there exists $m \in [n]$ with $p_j \geq m\ \forall j \in [m, n]$. Therefore,

$$\mathbf{x}^{\mathbf{p}} \notin I_S^{[\mathbf{n}]} \Leftrightarrow \begin{cases} \text{(i)} & p_l \leq l\ \forall l \in [n-1], \text{ and} \\ \text{(ii)} & \text{for any } m \in [n], \exists\ j \in [m, n] \text{ with } p_j < m. \end{cases}$$

As $p_n < n$, the Condition (i) is equivalent to $p_i \leq i \ \forall i$. Now we show that the condition (ii) follows from the condition (i). Let $m \in [n]$. If $p_m < m$, we can take $j = m$. So we assume that $p_m = m$ and the condition (ii) fails. Thus $p_j \geq m \ \forall j \in [m, n]$. As $p_i \leq i \ \forall i$, we see that $\{l \in [n] : p_l < m\} = [m-1]$, a contradiction to $|\{l \in [n] : p_l < m\}| \geq m$. Hence (i) implies (ii). □

Let Λ_n be the set of all parking functions of length n. Then $|\Lambda_n| = (n+1)^{n-1}$.

Definition 3.2. A parking function $\mathbf{p} = (p_1, \ldots, p_n) \in \Lambda_n$ is called a *Catalan parking function* if $p_i \leq i, \ \forall i \in [n]$.

Let Λ_n^{Cat} be the set of all Catalan parking functions of length n. Then in view of Lemma 3.1, $|\Lambda_n^{\text{Cat}}| = \dim_k \left(\frac{R}{I_S^{[\mathbf{n}]}} \right)$.

Proposition 3.3. *The number of standard monomials of* $\frac{R}{I_S^{[\mathbf{n}]}}$ *is given by*

$$\dim_k \left(\frac{R}{I_S^{[\mathbf{n}]}} \right) = n(n!) + \sum_{i=1}^{n} (-1)^i \sum_{0=j_0<j_1<\ldots<j_i<n} (n-j_i)(n-j_i)! \left(\prod_{q=1}^{i} (j_q - j_{q-1})! \right).$$

Proof. This proposition follows from a general result of Postnikov and Shapiro [9](Proposition 8.4). In fact,

$$\dim_k \left(\frac{R}{I_S^{[\mathbf{n}]}} \right) = \sum_{i=0}^{n} (-i)^i \sum_{C_1 \prec \ldots \prec C_i} \left(\prod_{q=0}^{i} (\prod_{j \in C_q - C_{q-1}} (\nu_{j,\{j\}} - \nu_{j,C_q})) \right) \left(\prod_{l \notin C_i} \nu_{l,\{l\}} \right),$$

where $C_0 = \emptyset$ and ν_{j,C_q} as in (2.1). A term in the above expression corresponding to a (strict) chain $C_1 \prec \ldots \prec C_i$ is zero if the chain has a singleton member. Thus the summation may be carried over chains of integer intervals of length i, which are determined by a sequence $0 = j_0 < j_1 < \ldots < j_i < n$ of positive integers on setting $C_t = [j_{i-t+1}, n]$. This completes the proof. □

Theorem 3.4. *Let* $A_{n+1} = [m_{ij}]_{(n+1)\times(n+1)}$, *where* $m_{ij} = (j-i+1)!$ *if* $i \leq j+1$ *and* $m_{ij} = 0$ *if* $i > j+1$. *Then* $\dim_k \left(\frac{R}{I_S^{[\mathbf{n}]}} \right) = (-1)^n \det(A_{n+1})$.

Proof. Let B be the matrix obtained by applying the row-operation $R_1 - R_2$ on $A = A_{n+1}$. Then $\det(B) = \det(A)$. The r^{th} column vector $\mathbf{v}_r$ of B is given by

$$\mathbf{v}_r = (r-1)(r-1)! e_1 + \sum_{s=1}^{r} (r-s)! e_{s+1} \quad \text{for} \quad 1 \leq r \leq n+1,$$

where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$ and $e_{n+2} = 0$. Since

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{n+1} = \det(B) e_1 \wedge \ldots \wedge e_{n+1} \tag{3.1}$$

by expanding the wedge product on the left hand side, we get the desired result in view of Proposition 3.3. In fact, for a sequence $0 = j_0 < j_1 < \ldots < j_i < n$, let $\mathbf{f}_r$ be a term from the vector $\mathbf{v}_r$ $(1 \leq r \leq n+1)$ given by

$$\mathbf{f}_r = \begin{cases} (n-j_i)(n-j_i)!e_1 & \text{if } r = n+1-j_i, \\ (j_{t+1}-j_t)!e_{n-j_{t+1}+2} & \text{if } r = n+1-j_t \ (t < i), \\ e_{r+1} & \text{if } r \neq n+1-j_t \ (0 \leq t \leq i). \end{cases}$$

Then $\mathbf{f}_1 \wedge \ldots \wedge \mathbf{f}_{n+1}$ equals

$$\left((n-j_i)(n-j_i)! \prod_{q=1}^{i} (j_q - j_{q-1})! \right) \left((-1)^{(n-j_i)} \prod_{q=1}^{i} (-1)^{j_q - j_{q-1} - 1} \right) e_1 \wedge \ldots \wedge e_{n+1}.$$

□

Now we consider the integer sequence (A003319) in OEIS [11]. The n^{th} term a_n of this sequence is the number of irreducible (or indecomposable) permutations of $[n] = \{1, 2, \ldots, n\}$. A permutation $\sigma \in \mathfrak{S}_n$ is *irreducible* if the restriction $\sigma|_{[j]}$ of σ to $[j]$ never induce a permutation of $[j]$ for any $1 \leq j < n$. It is easy to prove a recurrence relation $a_n = n! - \sum_{j=1}^{n-1} (j!) a_{n-j}, \quad n \geq 2$ with the initial condition $a_1 = 1$. As $(-1)^{n-1} \det(A_n)$ also satisfies the same recurrence relation, we have $a_n = (-1)^{n-1} \det(A_n)$. This shows that $|\Lambda_n^{\mathrm{Cat}}| = (-1)^n \det(A_{n+1}) = a_{n+1}$. As the number of Catalan parking functions of length n is same as the number of irreducible permutations of $[n+1]$, it would be an interesting problem to construct an explicit bijection between these objects.

4. Restricted Catalan parking functions

In this section, we study standard monomials of $\frac{R}{I_T^{[\mathbf{n}]}}$. Since $I_S^{[\mathbf{n}]} \subseteq I_T^{[\mathbf{n}]}$, every standard monomial of $\frac{R}{I_T^{[\mathbf{n}]}}$ is also a standard monomial of $\frac{R}{I_S^{[\mathbf{n}]}}$.

Definition 4.1. A Catalan parking function $\mathbf{p} = (p_1, \ldots, p_n) \in \Lambda_n$ is called a *restricted Catalan parking function* if for $i \in [n-1]$, either $p_i < i$ or $p_{i+1} < i$.

Let $\widetilde{\Lambda}_n^{\mathrm{Cat}}$ be the set of all restricted Catalan parking functions of length n. As in Lemma 3.1, we see that the standard monomials of $\frac{R}{I_T^{[\mathbf{n}]}}$ correspond bijectively to the restricted Catalan parking functions. Thus, $|\widetilde{\Lambda}_n^{\mathrm{Cat}}| = \dim_k \left(\frac{R}{I_T^{[\mathbf{n}]}} \right)$.

Using the minimal cellular resolution of $\frac{R}{I_T^{[\mathbf{n}]}}$ supported on the (labelled) order complex $\mathbf{\Delta}(\widetilde{\Sigma})$, the (fine) Hilbert series of $H\left(\frac{R}{I_T^{[\mathbf{n}]}}, \mathbf{x} \right)$ of $\frac{R}{I_T^{[\mathbf{n}]}}$ is easily calculated (see [4]). We have

$$H\left(\frac{R}{I_T^{[\mathbf{n}]}},\mathbf{x}\right)=\frac{\sum_{i=0}^{n}(-1)^i\sum_{(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}}\prod_{q=1}^{i}\left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}}x_j^{\mu_{j,\tilde{C}_q}}\right)}{(1-x_1)\cdots(1-x_n)},\tag{4.1}$$

where $\widetilde{\mathcal{F}}_{i-1}$ is the set of $i-1$-dimensional faces of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$, $(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}$ is a face represented by the (strict) chain $\tilde{C}_1\prec'\ldots\prec'\tilde{C}_i$ of length $i-1$, $\tilde{C}_0=\emptyset$ and $\mu_{j,\tilde{C}_q}$ is as in 2.2. Also, $H\left(\frac{R}{I_T^{[\mathbf{n}]}},\mathbf{x}\right)=\sum_{\mathbf{p}\in\widetilde{\Lambda}_n^{\mathrm{Cat}}}\mathbf{x}^{\mathbf{p}}$.

Proposition 4.2. *The number of standard monomials of* $\frac{R}{I_T^{[\mathbf{n}]}}$ *is given by*

$$\dim_k\left(\frac{R}{I_T^{[\mathbf{n}]}}\right)=\sum_{i=1}^{n}(-1)^{n-i}\sum_{\substack{(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}\\ \tilde{C}_1\cup\ldots\cup\tilde{C}_i=[n]}}\prod_{q=1}^{i}\left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}}\mu_{j,\tilde{C}_q}\right),$$

where summation is carried over all $i-1$*-dimensional faces* $(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}$ *of* $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ *with* $\bigcup_{l=1}^{i}\tilde{C}_l=[n]$ *and* $\tilde{C}_0=\emptyset$.

Proof. Clearly, $\dim_k\left(\frac{R}{I_T^{[\mathbf{n}]}}\right)=H\left(\frac{R}{I_T^{[\mathbf{n}]}},\mathbf{1}\right)$, where $\mathbf{1}=(1,\ldots,1)$. On the other hand, letting $\mathbf{x}\to\mathbf{1}$ in the rational function $H\left(\frac{R}{I_T^{[\mathbf{n}]}},\mathbf{x}\right)=\frac{Q(x)}{(1-x_1)\ldots(1-x_n)}$ given by 4.1 and applying L'Hopital's rule, we get

$$H\left(\frac{R}{I_T^{[\mathbf{n}]}},\mathbf{1}\right)=\frac{1}{(-1)^n}\frac{\partial^n Q(x)}{\partial x_1\ldots\partial x_n}|_{\mathbf{x}=\mathbf{1}}.$$

Now the term corresponding to a face $(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}$ is non-zero in the partial derivative $\frac{\partial^n Q(x)}{\partial x_1\ldots\partial x_n}$ only if $\tilde{C}_1\cup\ldots\cup\tilde{C}_i=[n]$. This completes the proof. $\square$

Remarks 4.3. (1) The (fine) Hilbert series of $H\left(\frac{R}{I_S^{[\mathbf{n}]}},\mathbf{x}\right)$ of $\frac{R}{I_S^{[\mathbf{n}]}}$ is given by

$$H\left(\frac{R}{I_S^{[\mathbf{n}]}},\mathbf{x}\right)=\frac{\sum_{i=0}^{n}(-1)^i\sum_{(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}}\prod_{q=1}^{i}\left(\prod_{j\in C_q-C_{q-1}}x_j^{\nu_{j,C_q}}\right)}{(1-x_1)\cdots(1-x_n)},$$

where $\mathcal{F}_{i-1}$ is the set of $i-1$-dimensional faces of $\mathbf{\Delta}(\Sigma_n)$, $(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}$ is a face represented by the (strict) chain $C_1\prec\ldots\prec C_i$ of length $i-1$, $C_0=\emptyset$ and ν_{j,C_q} is as in 2.1.

(2) Proceeding as in the proof of Proposition 4.2, we get

$$\dim_k\left(\frac{R}{I_S^{[\mathbf{n}]}}\right)=\sum_{i=1}^{n}(-1)^{n-i}\sum_{\substack{(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}\\ C_1\cup\ldots\cup C_i=[n]}}\prod_{q=1}^{i}\left(\prod_{j\in C_q-C_{q-1}}\nu_{j,C_q}\right),\tag{4.2}$$

where summation is carried over all $i-1$-dimensional faces $(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}$ of $\mathbf{\Delta}(\Sigma_n)$ with $\bigcup_{l=1}^{i}C_l=[n]$ and $C_0=\emptyset$. Since Proposition 3.3 is not immediate from formula 4.2, we used a result of Postnikov and Shapiro in its proof.

Let $b_n=|\widetilde{\Lambda}_n^{\mathrm{Cat}}|=\dim_k\left(\frac{R}{I_T^{[\mathbf{n}]}}\right)$ for $n\in\mathbb{N}$. Then $b_1=1,b_2=3$ and $b_3=11$.

Theorem 4.4. *The integer sequence* $\{b_n=|\widetilde{\Lambda}_n^{\mathrm{Cat}}|\}_{n=1}^{\infty}$ *satisfies a second-order recurrence relation*

$$b_n=nb_{n-1}+(n-1)b_{n-2};\quad n\geq 3$$

with initial conditions $b_1=1,b_2=3$.

Proof. From Proposition 4.2, $b_n=\sum_{i=1}^{n}(-1)^{n-i}\left(\sum_{\tilde{F}\in\widetilde{\mathcal{F}}_{i-1},\ \cup\tilde{F}=[n]}\pi(\tilde{F})\right)$, where summation is carried over $(i-1)$-dimensional faces $\tilde{F}=(\tilde{C}_1,\ldots,\tilde{C}_i)$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ with $\cup\tilde{F}=\tilde{C}_1\cup\ldots\cup\tilde{C}_i=[n]$ and $\pi(\tilde{F})=\prod_{q=1}^{i}\left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}}\mu_{j,\tilde{C}_q}\right)$. For $n\geq 3$, we divide such faces $\tilde{F}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ into three types.

1. A $(i-1)$-dimensional face $\tilde{F}$ is said to be of *Type-I* if the pair $(\tilde{C}_1,\tilde{C}_2)$ has one of the three values; namely, $(\{n-1\},[n-1,n]),([n,n+1],\{n-2\})$ or $([n,n+1],[n-2,n-1]$. On deleting $\tilde{C}_1$ from a $(i-1)$-dimensional face $\tilde{F}$ of Type-I, we get a $(i-2)$-dimensional face $\tilde{F}'$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-1})$ with $\cup\tilde{F}'=[n-1]$. Conversely, every such $(i-2)$ dimensional face $\tilde{F}'$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-1})$ extends uniquely to a $(i-1)$-dimensional face $\tilde{F}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ of Type-I. Also, for a Type-I face, we have $\pi(\tilde{F})=n\pi(\tilde{F}')$.

2. A $(i-1)$-dimensional face $\tilde{F}$ is said to be of *Type-II* if $\tilde{C}_1=[n-1,n]$. On deleting $\tilde{C}_1$ from a $(i-1)$-dimensional face $\tilde{F}$ of Type-II, we get a $(i-2)$-dimensional face $\tilde{F}''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ with $\cup\tilde{F}''=[n-2]$. Again, every such $(i-2)$ dimensional face $\tilde{F}''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ extends uniquely to a $(i-1)$-dimensional face $\tilde{F}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ of Type-II. Also, for a Type-II face, we have $\pi(\tilde{F})=(n-1)^2\pi(\tilde{F}'')$.

3. A $(i-1)$-dimensional face $\tilde{F}$ is said to be of *Type-III* if the pair $(\tilde{C}_1,\tilde{C}_2)=([n,n+1],[n-1,n])$. On deleting $\tilde{C}_1$ and $\tilde{C}_2$ from a $(i-1)$-dimensional face $\tilde{F}$ of Type-III, we get a $(i-3)$-dimensional face $\tilde{F}'''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ with $\cup\tilde{F}'''=[n-2]$. Again, every such $(i-3)$ dimensional face $\tilde{F}'''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ extends uniquely to a $(i-1)$-dimensional face $\tilde{F}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ of Type-III. Also, for a Type-III face, we have $\pi(\tilde{F})=n(n-1)\pi(\tilde{F}''')$.

Now dividing the summation in b_n according to the Type of $i-1$-dimensional faces, we get

$$b_n = \sum_{i=1}^{n} (-1)^{n-i} \left[\sum_{\tilde{F} \text{ Type-I}} + \sum_{\tilde{F} \text{ Type-II}} + \sum_{\tilde{F} \text{ Type-III}} \right] \pi(\tilde{F}).$$

As $n-i = (n-1)-(i-1) = (n-2)-(i-1)+1 = (n-2)-(i-2)$, we clearly have $b_n = nb_{n-1} + [-(n-1)^2 + n(n-1)]b_{n-2}$. □

We consider the integer sequence (A000255) in OEIS [11]. The n^{th} term $\tilde{a}_n$ of this sequence counts permutations of $[n+1]$ having no substring $\{l, l+1\}$. It is known that for $n \geq 1$, $\tilde{a}_n = \det([\tilde{m}_{ij}]_{n\times n})$, where $\tilde{m}_{ii} = \tilde{m}_{i+1i} = i, \tilde{m}_{ii+1} = -1$ and $m_{ij} = 0$ if $|i-j| \geq 2$. It is straight forward to check that the integer sequence $\{\tilde{a}_n\}_{n=1}^{\infty}$ satisfies the second-order recurrence relation $\tilde{a}_n = n\tilde{a}_{n-1} + (n-1)\tilde{a}_{n-2};\ n \geq 3$ with initial conditions $\tilde{a}_1 = 1, \tilde{a}_2 = 3$.

Theorem 4.5.

$$|\widetilde{\Lambda}_n^{\mathrm{Cat}}| = \dim_k \left(\frac{R}{I_T^{[\mathbf{n}]}} \right) = \det\left([\tilde{m}_{ij}]_{n\times n}\right).$$

Proof. Since both integer sequences $\{b_n = |\widetilde{\Lambda}_n^{\mathrm{Cat}}|\}_{n=1}^{\infty}$ and $\{\tilde{a}_n = \det([\tilde{m}_{ij}]_{n\times n})\}_{n=1}^{\infty}$ satisfy the same second-order recurrence relation with the same initial conditions, we have $b_n = \tilde{a}_n,\ \forall n \geq 1$. □

5. Some generalizations

All the results about monomial ideals I_S, I_T and their Alexander duals can be extended to a slightly larger class of monomial ideals. In this section, we outline these generalizations. Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $1 \leq u_1 < \ldots < u_n$ and for every $\sigma \in \mathfrak{S}_n$, $\mathbf{x}^{\sigma\mathbf{u}} = \prod_{i=1}^{n} x_i^{u_{\sigma(i)}}$ be the associated monomial. We consider the monomial ideals $I_S(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in S \rangle$ and $I_T(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in T \rangle$ in R. Clearly, $I_S((1,2,\ldots,n)) = I_S$ and $I_T((1,2,\ldots,n)) = I_T$. The monomial ideal $I(\mathbf{u}) = I_{\mathfrak{S}_n}(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in \mathfrak{S}_n \rangle$ is again called a *permutohedron ideal.*

For an integer $c \geq 1$, set $\mathbf{u_n} + \mathbf{c} - \mathbf{1} = (u_n + c - 1, \ldots, u_n + c - 1) \in \mathbb{N}^n$. We consider the Alexander dual $I_S(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ (or $I_T(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$) of $I_S(\mathbf{u})$ (or $I_T(\mathbf{u})$) with respect to $\mathbf{u_n} + \mathbf{c} - \mathbf{1}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = u_n - u_i + c$.

Lemma 5.1. *The minimal generators of the Alexander duals* $I_S(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ *and* $I_T(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ *are given by*

$$I_S(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} = \left\langle x_l^{\lambda_{n-l}}, \left(\prod_{j=m}^{n} x_j \right)^{\lambda_{n-m+1}} : 1 \leq l \leq n-1, 1 \leq m \leq n \right\rangle$$

and

$$I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]} = \left\langle x_l^{\lambda_{n-l}}, \left(\prod_{j\in[m,m+1]} x_j\right)^{\lambda_{n-m+1}} : 1 \leq l \leq n-1, 1 \leq m \leq n \right\rangle,$$

where $[m, m+1] = \{m, m+1\}$ *for* $m \in [n-1]$ *and* $[n, n+1] = \{n\}$.

Proof. Proceeding as in the proof of Lemma 2.1 and 2.2, we get the minimal generators on taking $\mathbf{b}_l(\mathbf{u}) = (u_n+c-1, \dots, u_{n-l}-1, \dots, u_n+c-1)$, $\mathbf{b}_{[m,n]}(\mathbf{u}) = (u_n+c-1, \dots u_n+c-1, u_{n-m+1}-1, \dots, u_{n-m+1}-1)$ and $\mathbf{b}_{[m,m+1]}(\mathbf{u}) = (u_n+c-1, \dots, u_{n-m+1}-1, u_{n-m+1}-1, \dots, u_n+c-1)$, in place of $\mathbf{b}_l, \mathbf{b}_{[m,n]}$ and $\mathbf{b}_{[m,m+1]}$, respectively. □

Remark 5.2. Since we are interested in the Alexander duals $I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ and $I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ such that their respective quotients $\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ and $\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ are Artinian, we have assumed that $u_1 \geq 1$. However, both the ideals $I_S(\mathbf{u})$ and $I_T(\mathbf{u})$ are also defined for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$ with $u_1 = 0$.

We label the order complexes $\Delta(\Sigma_n)$ and $\Delta(\widetilde{\Sigma}_n)$ so that the monomial ideals generated by vertex labels are $I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ and $I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}$, respectively. If F is an $i-1$-dimensional face of $\mathbf{\Delta}(\Sigma_n)$ corresponding to a (strict) chain $C_1 \prec \dots \prec C_i$ of length $i-1$ in Σ_n, then the monomial label $\mathbf{x}^{\nu^{\mathbf{u}}(F)}$ on F is given by

$$\mathbf{x}^{\nu^{\mathbf{u}}(F)} = \prod_{q=1}^{i} \left(\prod_{j\in C_q - C_{q-1}} x_j^{\nu^{\mathbf{u}}_{j,C_q}} \right),$$

where $C_0 = \emptyset$ and

$$\nu^{\mathbf{u}}_{j,C_q} = \begin{cases} \lambda_{n-l} & \text{if } C_q = \{l\}, \\ \lambda_{n-m+1} & \text{if } C_q = [m,n]. \end{cases} \tag{5.1}$$

Similarly, if $\widetilde{F}$ is an $i-1$-dimensional face of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ corresponding to a (strict) chain $\tilde{C}_1 \prec' \dots \prec' \tilde{C}_i$ of length $i-1$ in $\widetilde{\Sigma}_n$, then the monomial label $\mathbf{x}^{\mu^{\mathbf{u}}(\widetilde{F})}$ on $\widetilde{F}$ is given by

$$\mathbf{x}^{\mu^{\mathbf{u}}(\widetilde{F})} = \prod_{q=1}^{i} \left(\prod_{j\in \tilde{C}_q - \tilde{C}_{q-1}} x_j^{\mu^{\mathbf{u}}_{j,\tilde{C}_q}} \right),$$

where $\tilde{C}_0 = \emptyset$ and

$$\mu^{\mathbf{u}}_{j,\tilde{C}_q} = \begin{cases} \lambda_{n-l} & \text{if } \tilde{C}_q = \{l\}, \\ \lambda_{n-m+1} & \text{if } \tilde{C}_q = [m,m+1]. \end{cases} \tag{5.2}$$

Now we have the following generalization of Theorem 2.7.

Proposition 5.3. *For* $0 \leq r \leq n-1$,

$$\beta_r\left(I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}\right) = f_r(\mathbf{\Delta}(\Sigma_n)) \quad \text{and} \quad \beta_r\left(I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}\right) = f_r(\mathbf{\Delta}(\widetilde{\Sigma}_n)).$$

Proof. Both $I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ and $I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ are order monomial ideals, thus the cellular resolution supported on the order complexes $\mathbf{\Delta}(\Sigma_n)$ and $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ give their minimal resolutions, respectively. □

Remarks 5.4. (1) $I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ and $I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ are both strongly generic ideals. (2) The LCM-lattices of $I_S^{[\mathbf{n}]}$ and $I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ (or $I_T^{[\mathbf{n}]}$ and $I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}$) are isomorphic by an isomorphism induced by 'relabeling' [3]. This also establishes the equality of Betti numbers

$$\beta_r\left(I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}\right) = \beta_r\left(I_S^{[\mathbf{n}]}\right) \quad \text{and} \quad \beta_r\left(I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}\right) = \beta_r\left(I_T^{[\mathbf{n}]}\right).$$

We recall that the standard monomials of $\frac{R}{I(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ are of the form $\mathbf{x^p}$, where $\mathbf{p}$ is a λ-parking function of length n for $\lambda = (\lambda_1, \ldots, \lambda_n)$; $\lambda_i = u_n - u_i + c$. Now the standard monomials of $\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ and $\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ are given as follows.

Lemma 5.5. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *be a* λ*-parking function of length* n*. Then*

a) $\mathbf{x^p} \notin I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]} \Leftrightarrow p_j < \lambda_{n-j}\ \forall j \in [n-1]$.

b) $\mathbf{x^p} \notin I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]} \Leftrightarrow p_j < \lambda_{n-j}\ \forall j \in [n-1]$ *and either* $p_j < \lambda_{n-j+1}$ *or* $p_{j+1} < \lambda_{n-j+1}$.

Proof. These conditions are verified as in the proof of Lemma 3.1. □

Definition 5.6. A λ-parking function $\mathbf{p} = (p_1, \ldots, p_n)$ of length n is said to be a *Catalan* λ*-parking function* if $p_j < \lambda_{n-j}\ \forall j \in [n-1]$. Also, a Catalan λ-parking function $\mathbf{p} = (p_1, \ldots, p_n)$ is said to be a *restricted Catalan* λ*-parking function* if in addition, either $p_j < \lambda_{n-j+1}$ or $p_{j+1} < \lambda_{n-j+1}\ \forall j \in [n-1]$.

Henceforth, we take $\mathbf{u} = (u_1, \ldots, u_n)$ such that $u_1 \geq 1$ and $u_i = u_1 + (i-1)b$ for some integer $b \geq 1$. In other words, the sequence $\{u_i\}$ is an arithmetic progression. The sequence $\{\lambda_i\}$ with $\lambda_i = u_n - u_i + c = c + (n-i)b\ \forall i \in [n]$ is also an arithmetic progression. Sometimes, we put $\lambda_0 = c + nb$. To emphasize that λ depends only on b and c, we write $\lambda = \lambda(c,b)$. Let $\Lambda_n(\lambda(c,b))$ be the set of $\lambda(c,b)$-parking functions of length n and its subset consisting of Catalan $\lambda(c,b)$-parking functions (or restricted Catalan $\lambda(c,b)$-parking functions) be denoted by $\Lambda_n^{\mathrm{Cat}}(\lambda(c,b))$ (or $\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))$). Then $|\Lambda_n(\lambda(c,b))| = c(c+nb)^{n-1}$ (see [8, 9]). In view of Lemma 5.5, we have $|\Lambda_n^{\mathrm{Cat}}(\lambda(c,b))| = \dim_k\left(\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$ and $|\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))| = \dim_k\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$.

Theorem 5.7. *Let $\mathbf{u} = (u_1, \dots, u_n)$ with $u_1 \geq 1$ and $u_i = u_1 + (i-1)b \ \forall i \in [n]$.*
(1) The number of standard monomials of $\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ is given by

$$\dim_k\left(\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \lambda_1 \prod_{t=1}^{n-1} \lambda_t + \sum_{i=1}^{n} (-1)^{n-i} \sum_{0=j_0<j_1<\ldots<j_i<n} \Theta(j_1, \dots, j_i),$$

where summation runs over all sequences $0 < j_1 < \ldots < j_i < n$ and

$$\Theta(j_1, \dots, j_i) = b^{n-j_i+1}(n-j_i)(n-j_i)! \left(\prod_{q=2}^{i} b^{j_q - j_{q-1}}(j_q - j_{q-1})!\right) \prod_{s=n-j_1+1}^{n-1} \lambda_s.$$

(2) Let $A_{n+1}^{\lambda} = \left[m_{ij}^{\lambda}\right]_{(n+1)\times(n+1)}$ be a matrix such that

$$m_{ij}^{\lambda} = \begin{cases} b^{j-i+1}(j-i+1)! & \text{if } i \leq j+1; j < n+1, \\ 0 & \text{if } i > j+1; j < n+1, \\ \prod_{s=i-1}^{n-1} \lambda_s & \text{if } j = n+1. \end{cases}$$

Then $|\Lambda_n^{\mathrm{Cat}}(\lambda(c,b))| = \dim_k\left(\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = (-1)^n \det\left(A_{n+1}^{\lambda}\right)$.

Proof. Proceeding as in the proof of Proposition 3.3, we get an expression for $\dim_k\left(\frac{R}{I_S(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$ exactly similar to that of $\dim_k\left(\frac{R}{I_S^{[\mathbf{n}]}}\right)$, with $\nu_{j,C_q}^{\mathbf{u}}$ in place of ν_{j,C_q}. Now a straight forward calculation verifies the first part. On applying the row operation $R_1 - bR_2$ on the matrix A_{n+1}^{λ} and expanding the determinant of the resulting matrix along the $(n+1)$th column, we also get the second part. ☐

The (fine) Hilbert series $H\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}, \mathbf{x}\right)$ of $\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ is obtained from 4.1 by simply replacing $\mu_{j,\tilde{C}_q}$ with $\mu_{j,\tilde{C}_q}^{\mathbf{u}}$ (as in 5.2). Thus

$$H\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}, \mathbf{x}\right) = \frac{\sum_{i=0}^{n}(-1)^i \sum_{(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1}} \prod_{q=1}^{i}\left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}} x_j^{\mu_{j,\tilde{C}_q}^{\mathbf{u}}}\right)}{(1-x_1)\cdots(1-x_n)}.$$

Proposition 5.8. *The number of standard monomials of $\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ is given by*

$$\dim_k\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \sum_{i=1}^{n}(-1)^{n-i} \sum_{\substack{(\tilde{C}_1,\ldots,\tilde{C}_i)\in\widetilde{\mathcal{F}}_{i-1} \\ \tilde{C}_1\cup\ldots\cup\tilde{C}_i=[n]}} \prod_{q=1}^{i}\left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}} \mu_{j,\tilde{C}_q}^{\mathbf{u}}\right),$$

where summation is carried over all $i-1$-dimensional faces $(\tilde{C}_1, \dots, \tilde{C}_i) \in \widetilde{\mathcal{F}}_{i-1}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ with $\bigcup_{l=1}^{i} \tilde{C}_l = [n]$ and $\tilde{C}_0 = \emptyset$.

Proof. Proceed as in the proof of Proposition 4.2. □

For an integer $n \geq 1$, let $b_n^\lambda = |\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))| = \dim_k\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$. Then $b_1^\lambda = c$ and $b_2^\lambda = c(c+2b)$.

Theorem 5.9. *The integer sequence* $\{b_n^\lambda = |\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))|\}_{n=1}^\infty$ *satisfies a second-order recurrence relation*

$$b_n^\lambda = ((n-1)b+c)\, b_{n-1}^\lambda + \left((n-2)b^2+bc\right) b_{n-2}^\lambda; \quad n \geq 3$$

with initial conditions $b_1^\lambda = c, b_2^\lambda = c(c+2b)$.

Proof. From Proposition 5.8, $b_n^\lambda = \sum_{i=1}^n (-1)^{n-i}\left(\sum_{\tilde{F}\in\mathcal{F}_{i-1},\ \cup\tilde{F}=[n]} \pi^{\mathbf{u}}(\tilde{F})\right)$, where summation is carried over $(i-1)$-dimensional faces $\tilde{F} = (\tilde{C}_1,\ldots,\tilde{C}_i)$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ with $\cup\tilde{F} = \tilde{C}_1 \cup \ldots \cup \tilde{C}_i = [n]$ and $\pi^{\mathbf{u}}(\tilde{F}) = \prod_{q=1}^i \left(\prod_{j\in\tilde{C}_q-\tilde{C}_{q-1}} \mu_{j,\tilde{C}_q}^{\mathbf{u}}\right)$. For $n \geq 3$, we divide such faces $\tilde{F}$ of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$ into three types as in the proof of Theorem 4.4.

Let $\tilde{F}$ be an $(i-1)$-dimensional face of $\mathbf{\Delta}(\widetilde{\Sigma}_n)$. If $\tilde{F}$ is of *Type-I*, then there is a unique $(i-2)$-dimensional face $\tilde{F}'$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-1})$ with $\cup\tilde{F}' = [n-1]$ and $\pi^{\mathbf{u}}(\tilde{F}) = \lambda_1\pi^{\mathbf{u}}(\tilde{F}')$. If $\tilde{F}$ is of *Type-II*, then there is a unique $(i-2)$-dimensional face $\tilde{F}''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ with $\cup\tilde{F}'' = [n-2]$ and $\pi^{\mathbf{u}}(\tilde{F}) = (\lambda_2)^2\pi^{\mathbf{u}}(\tilde{F}'')$. Again, if $\tilde{F}$ is of *Type-III*, then there is a unique $(i-3)$-dimensional face $\tilde{F}'''$ of $\mathbf{\Delta}(\widetilde{\Sigma}_{n-2})$ with $\cup\tilde{F}''' = [n-2]$ and $\pi^{\mathbf{u}}(\tilde{F}) = \lambda_1\lambda_2\pi^{\mathbf{u}}(\tilde{F}''')$.

Now rearranging terms in b_n^λ, we get

$$b_n^\lambda = \sum_{i=1}^n (-1)^{n-i}\left[\sum_{\tilde{F}\ \mathrm{Type-I}} + \sum_{\tilde{F}\ \mathrm{Type-II}} + \sum_{\tilde{F}\ \mathrm{Type-III}}\right]\pi^{\mathbf{u}}(\tilde{F}).$$

As $n-i = (n-1)-(i-1) = (n-2)-(i-1)+1 = (n-2)-(i-2)$, we clearly have $b_n^\lambda = \lambda_1 b_{n-1}^\lambda + [-(\lambda_2)^2 + \lambda_1\lambda_2]b_{n-2}^\lambda$. □

Let $\lambda = \lambda(c,b)$ and let $\left[\widetilde{m}_{ij}^\lambda\right]_{n\times n}$ be a tridiagonal matrix such that

$$\widetilde{m}_{ij}^\lambda = \begin{cases} c+(i-1)b & \text{if } i=j \text{ or } i=j+1, \\ -b & \text{if } j=i+1, \\ 0 & \text{if } |i-j|\geq 2. \end{cases}$$

Theorem 5.10.

$$|\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))| = \dim_k\left(\frac{R}{I_T(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \det\left([\tilde{m}_{ij}^\lambda]_{n\times n}\right).$$

Proof. Since integer sequences $\{b_n^\lambda = |\widetilde{\Lambda}_n^{\mathrm{Cat}}(\lambda(c,b))|\}_{n=1}^\infty$ and $\{\det\left([\tilde{m}_{ij}^\lambda]_{n\times n}\right)\}_{n=1}^\infty$ satisfy the same second-order recurrence relation with the same initial conditions, they must be identical. □

Acknowledgements : Thanks are due to the anonymous referee for many valuable suggestions that improved overall presentation of the paper.

References

[1] Bayer D., Peeva I. and Sturmfels B., *Monomial resolutions*, Mathematical Research Letters **5** (1998), 31-46.

[2] Bayer D. and Sturmfels B., *Cellular resolution of monomial modules*, Journal für die Reine und Angewandte Mathematik **502** (1998), 123-140.

[3] Gasharov V., Peeva I. and Welker V., *The lcm-lattice in monomial resolutions*, Math. Res. Lett. **5-6** (1999), 521-532.

[4] Kumar A. and Kumar C., *Alexander duals of multipermutohedron ideals*, Proc. Indian Acad. Sci.(Math Sci.) Vol.**124**, No.1, February 2014, 1-15.

[5] Miller E., Sturmfels B. and Yanagawa K., *Generic and cogeneric monomial ideals*, J. Symbolic Comput. **29** (2000), 691-708.

[6] Miller E. and Sturmfels B., *Combinatorial commutative algebra*, Graduate Texts in Mathematics Vol **227**, Springer 2004.

[7] Miller E., *Alexander duality for monomial ideals and their resolutions*, Rejecta Mathematica **1**, no 1, (2009), 18-57.

[8] Pitman J. and Stanley R., *A polytope related to empirical distribution, plane Trees, parking functions and the associahedron*, Discrete and Computational Geometry **27** (2002), 603-634.

[9] Postnikov A. and Shapiro B., *Trees, parking functions, syzygies, and deformation of Monomial ideals*, Trans. Amer. Math. Soc. **356**, (2004), 3109-3142.

[10] Simion R. and Schmidt F. W., *Restricted permutations*, Europ. J. Combinatorics **6** (1985), 383-406.

[11] Sloane N. J. A., *On-line encyclopedia of integer sequences*, http://www.research.att.com/˜njas/sequences/

[12] Yan C. H., *On the enumeration of generalized parking functions*, Proceedings of the 31-st Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000), Congressus Numerantium **147** (2000), 201-209.

DAV University, Jalandhar, Punjab - 144 012, India
E-mail address: `ajaychhabra.msc@gmail.com`

IISER Mohali, Knowledge City, Sector 81, SAS Nagar, Punjab -140 306, India.
E-mail address: `chanchal@iisermohali.ac.in`