The theory of Suslin matrices

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Abstract

I shall describe Suslin's construction of the Suslin matrices, his need to consider these matrices, their properties, and the theory of Suslin matrices that I jointly developed with Selby Jose. I will give some applications, and state a few open questions regarding these matrices.

Examples of Suslin matrices

I shall describe Suslin's construction of the Suslin matrices, his need to consider these matrices, their properties, and the theory of Suslin matrices that I jointly developed with Selby Jose.

The 2×2 **case:**

$$\begin{pmatrix} a_0 & a_1 \\ -b_1 & b_0 \end{pmatrix}$$

$$S_1((a_0, a_1), (b_0, b_1))$$

The 4×4 **case:**

$$\begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix}$$

$$S_2((a_0, a_1, a_2), (b_0, b_1, b_2))$$

The 8×8 case

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_0 & 0 & 0 & 0 & a_1 & -b_3 & b_2 \\ 0 & 0 & a_0 & 0 & -b_2 & a_3 & b_1 & 0 \\ 0 & 0 & 0 & a_0 & -b_3 & -a_2 & 0 & b_1 \\ -b_1 & 0 & a_2 & a_3 & b_0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & b_2 & 0 & b_0 & 0 & 0 \\ -b_2 & a_3 & -a_1 & 0 & 0 & 0 & b_0 & 0 \\ -b_3 & -a_2 & 0 & -a_1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$S_3((a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3))$$

$$= \begin{pmatrix} a_0 I_4 & S_2((a_1, a_2, a_3), (b_1, b_2, b_3)) \\ -S_2((b_1, b_2, b_3), (a_1, a_2, a_3))^t & b_0 I_4 \end{pmatrix}$$

The Suslin matrix $S_r(v, w)$

The construction of the Suslin matrix $S_r(v, w)$ is possible once we have two rows v, w.
The matrix will be invertible if the dot product $\langle v, w\rangle = v \cdot w^t = 1$.
(The rows are then automatically **unimodular rows**.)
A. Suslin's inductive definition: Let

$$v = (a_0, a_1, \cdots, a_r) = (a_0, v_1),$$

with $v_1 = (a_1, \cdots, a_r)$,

$$w = (b_0, b_1, \cdots, b_r) = (b_0, w_1),$$

with $w_1 = (b_1, \cdots, b_r)$. Set $S_0(v, w) = a_0$, and set

$$S_r(v, w) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}.$$

The Suslin matrices

A. Suslin noted that

$$\begin{aligned} S_r(v,w)S_r(w,v)^t &= (v \cdot w^t) I_{2^r} \\ &= S_r(w,v)^t S_r(v,w), \end{aligned}$$

and $\det S_r(v,w) = (v \cdot w^t)^{2^{r-1}}$, for $r \geq 1$.

The “Square Unimodular Rows’

Swan-Towber realized in 1974 that the unimodular “square” rows (a^2, b, c) can always be completed to an invertible matrix. We write two completions below.

$$aa' + bb' + cc' = 1$$

$$\begin{pmatrix} a^2 & b & c \\ b + ac' & -c'^{\,2} + ba'c' & -a' + b'c' - c'bb' \\ c - ab' & a' + b'c' + a'cc' & -b'^{\,2} - a'b'c \end{pmatrix}$$

Swan-Towber

We will denote this matrix by $ST_2((a, b, c), (a', b', c'))$.

Krusemeyer-Suslin completion

Mark Kruesemeyer and A. Suslin independently tried to explain the Swan–Towber result; and came up with the following completion.

$$\begin{pmatrix} a^2 & b & c \\ -b-2ac' & c'^{\,2} & a'-b'c' \\ -c+2ab' & -a'-b'c' & b'^{\,2} \end{pmatrix}$$

$$\det = (aa'+bb'+cc')^2$$

M. Krusemeyer & A. Suslin

The factorial row

Suslin had a larger vision in mind: The Suslin matrices were introduced by A. Suslin to show that a unimodular row of the form $(a_0, a_1, a_2^2, \cdots, a_r^r)$ can be completed to an invertible matrix $\beta_r(v, w)$ of determinant one, where $v = (a_0, a_1, \ldots, a_r)$, $w = (b_0, b_1, \ldots, b_4)$, with $\langle v, w \rangle = 1$.
The actual completion can be got by doing a series of row and column operations to the matrix $S_r(v, w)$ to reduce it to size $(r + 1)$.
Nothing much is known about the relation between $ST_2((a, b, c), (a', b', c'))$ and the Krusemeyer–Suslin completion. Such relations would throw more light on the study of projective modules and the study of alternating matrices.

Use of Suslin matrices: A sample

• A. Suslin used the Suslin matrices to show that a stably free projective module of rank d over an affine algebra of dimension d over an algebraically closed field is a free module.

• A. Suslin showed that

$$\mathrm{SK}_1\left(\frac{k[x_1,\cdots,x_n,y_1,\cdots,y_n]}{(\sum_{i=1}^n x_iy_i-1)}\right)\simeq\mathbb{Z},$$

with generator $[S_{n-1}((x_1,\cdots,x_n),(y_1,\cdots,y_n))]$.

• Let $\sum_{i=1}^{n} x_iy_i = 1$. Let P, P^* be the projective modules corresponding to the unimodular rows $(x_1,\cdots,x_n)$, $(y_1,\cdots,y_n)$. Then $P^* \simeq \mathrm{Hom}_R(P,R)$, the dual of P.

If n is even then $P \simeq P^*$. However, if $n > 1$ is odd then M.V. Nori, and R.G. Swan independently showed (using topological arguments) that P, P^* need not be isomorphic. This can also be shown using the Suslin matrices if $n > 3$.

• M. Boratynski showed that any ideal I in a polynomial ring R over a field can be generated upto radical by $m = \mu\left(\frac{I}{I^2}\right)$ elements, i.e. $\sqrt{I} = \sqrt{(f_1, \cdots, f_m)}$, for some $f_1, \cdots, f_m \in R$.

• R.A. Rao showed that unimodular polynomial rows over a commutative local ring of dimension three can be completed to an invertible matrix if $6R = R$.

• With my students, I used the Suslin matrices to show that the injective stabilization for a non-singular affine algebra A over an algebraically closed field, for the symplectic $K_1Sp(A)$, falls to $d + 1$ (from $2d + 4$, where d is the dimension of A. I also used these matrices to show that in general, the injective stabilization bounds for the orthogonal $K_1O(A)$ groups will not fall in general, even for smooth affine algebras over an algebraically closed field.

The Suslin forms

To understand the beautiful Suslin matrices we recall A. Suslin's sequence of forms $J_r \in M_{2^r}(R)$ given by the recurrence formulae:

$$J_r = \begin{cases} 1 & \text{for r} = 0 \\ J_{r-1} \perp -J_{r-1}, & \text{for r even} \\ J_{r-1} \top - J_{r-1}, & \text{for r odd.} \end{cases},$$

(Here $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, while $\alpha \top \beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$).
It is easy to see that $\det J_r = 1$, for all r, and that

$$J_r^t = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r.$$

Moreover, J_r is antisymmetric if $r = 4k+1$ and $r = 4k+2$, whereas J_r is symmetric for $r = 4k$ and $r = 4k+3$.

The Suslin identities

Suslin noted that the following formulae are valid:

$$\begin{aligned}
\text{for } r=4k : (S_r(v,w)J_r)^t &= S_r(v,w)J_r; \\
\text{for } r=4k+1 : S_r(v,w)J_rS_r(v,w)^t &= \langle v.w\rangle J_r; \\
\text{for } r=4k+2 : (S_r(v,w)J_r)^t &= -S_r(v,w)J_r; \\
\text{for } r=4k+3 : S_r(v,w)J_rS_r(v,w)^t &= \langle v,w\rangle J_r.
\end{aligned}$$

The above identities show that the Suslin matrix $S_r(v,w)$ is **symplectic** if $r = 1$ modulo (4), and **orthogonal** if $r = 3$ modulo (4).

The Fundamental Property of Suslin Matrices

Let

$$\begin{aligned} v &= (a_0, a_1, \ldots, a_r) = (a_0, v_1), \\ w &= (b_0, b_1, \ldots, b_r) = (b_0, w_1), \\ s &= (c_0, c_1, \ldots, c_r) = (c_0, s_1), \\ t &= (d_0, d_1, \ldots, d_r) = (d_0, t_1), \end{aligned}$$

for some v_1, w_1, s_1, $t_1 \in M_{1,r}(R)$ below. Then

$$S_r(s,t)S_r(v,w)S_r(s,t) = S_r(v', w') \quad (1)$$

$$S_r(t,s)S_r(w,v)S_r(t,s) = S_r(w', v'), \quad (2)$$

for some $v', w' \in M_{1r+1}(R)$, which depend linearly on v, w and quadratically on s, t. Consequently, $v' \cdot {w'}^T = (s \cdot t^T)^2 (v \cdot w^T)$.

The Fundamental property made one realize that there is a Clifford algebra context in which the Suslin matrices can be studied. This was recently done by my student **Vineeth Chintala**.

A natural involution

The case when r is EVEN.

Let $\alpha = \prod_{i=1}^{n} S_i$ be a product of Suslin matries $S_i = S_r(v_i, w_i)$, and let α^* denote $\prod_{i=n}^{1} S_i$. If r is even, then $\alpha \mapsto \alpha^*$ is a well defined anti-involution on the subgroup of $\mathrm{SL}_{2^r}(R)$ generated by all the Suslin matrices $S_r(v, w)$, $\langle v, w \rangle = 1$. This group is denoted by $\mathrm{SUm}_r(R)$.
By Suslin's identities,

$$S_r(v, w) = J_r S_r(v, w)^T J_r^{-1}.$$

Hence, $\alpha^* = J_r \alpha^T J_r^{-1}$, and we are done.
As a consequence, the Fundamental property of Suslin matrices enables one to define an action of the group $\mathrm{SUm}_r(R)$ on the Suslin vector space (which is the space of all Suslin matrices) when r is **even**.
We analyse this action next.

Suslin matrices and Orthogonal transformations

Corollary: The above action enables one to associate a linear transformation T_g of the Suslin space with a Suslin matrix g, via

$$T_g(x, y) = (x', y'),$$

where $gS_r(x, y)g^* = S_r(x', y')$.
Moreover, if g is a product of Suslin matrices $S_r(v_i, w_i)$, with $\langle v_i, w_i \rangle = 1$, for all i, then

$$T_g \in O_{2(r+1)}(R),$$

i.e.

$$\begin{aligned} \langle T_g(v, w), T_g(s, t) \rangle &= \langle (v, w), (s, t) \rangle \\ &= \langle v, w \rangle + \langle s, t \rangle. \end{aligned}$$

Translating the Fundamental identities

Corollary: The above action induces a canonical homomorphism

$$\varphi : \mathrm{SUm}_r(R) \to \mathrm{SO}_{2(r+1)}(R),$$

with

$$\varphi(S_r(v,w)) \;=\; T_{S_r(v,w)} = \tau_{(v,w)} \circ \tau_{(e_1 e_1)},$$

where $\tau_{(v,w)}$ is the standard reflection with respect to the vector $(v,w) \in R^{2(r+1)}$ (of length one) given by the formula

$$\tau_{(v,w)}(s,t) \;=\; \langle v,w\rangle(s,t) - (\langle v,t\rangle + \langle s,w\rangle)(v,w).$$

Proof: This is easy to verify, via an application of the earlier Corollary, and the Fundamental identities of Suslin matrices, once it is pointed out.

The quotient groups

We compute the kernel of the map

$$\varphi : \mathrm{SUm}_r(R) \quad \rightarrow \quad \mathrm{SO}_{2(r+1)}(R),$$

and show that it consists of scalars uI_{2^r}, with $u^2 = 1$. This follows from:

Lemma: Let R be a commutative ring in which 2 is invertible. Let $\alpha \in \mathrm{SUm}_r(R)$. Suppose that $\alpha S_r(v, w)\alpha^* = S_r(v, w)$, for all $S_r(v, w) \in \mathrm{EUm}_r(R)^*$. Then α^* centralizes $\mathrm{EUm}_r(R)^*$. Consequently, α is a scalar uI_{2^r}, with $u^2 = 1$.

Moreover, we show

Theorem: There is an injection of groups

$$\mathrm{SUm}_r(R)/\mathrm{EUm}_r(R) \quad \hookrightarrow \quad \mathrm{SO}_{2(r+1)}(R)/\mathrm{EO}_{2(r+1)}(R),$$

where $\mathrm{EUm}_r(R)$ is the subgroup of $\mathrm{SUm}_r(R)$ generated by the Suslin matrices

$$\{S_r(e_1\varepsilon, e_1\varepsilon^{t^{-1}}) \mid \varepsilon \in \mathrm{E}_{r+1}(R)\}.$$

Dual is not isomorphic

Let $\sum_{i=1}^{n} x_i y_i = 1$.
Let P, P^* be the projective modules corresponding to the unimodular rows

$$(x_1, \cdots, x_n), (y_1, \cdots, y_n).$$

Then $P^* \simeq \mathrm{Hom}_R(P, R)$, the dual of P.
If n is even then $P \simeq P^*$. However, if $n > 1$ is odd then M.V. Nori, and R.G. Swan independently showed (using topological arguments) that P, P^* need not be isomorphic. This can also be shown using the Suslin matrices if $n > 3$.

Proof

Suppose that $v\sigma = w$, for some $\sigma \in GL_{2n-1}(R)$. Then

$$\begin{aligned} \mathrm{wt}\ (w) &= \mathrm{wt}\ (v\sigma) \\ &= \mathrm{wt}\ (v) + \sum_{i=0}^{2_{n-1}} (-1)^i [\wedge^i \sigma]. \end{aligned}$$

Since $SK_1(R) = \mathbb{Z}$, $[\sigma] = [S(v, w)]^r$, for some r. Hence, $[\wedge^i \sigma] = r[\wedge^i S(v_1 w)]$. Therefore,

$$\begin{aligned} \sum_{i=0}^{2_{n-1}} (-1)^i [\wedge^i \sigma] &= r \sum_{i=0}^{2n-1} (-1)^i [\wedge^i S(v, w)] \\ &= r\mathbf{wt}\ (\overline{x}_1, \overline{x}_2, \overline{x}_3^{-2}, \cdots, \overline{x}_{2n-1}^{2n-2}) \\ &= r(2n-2)!\mathrm{wt}\ (v). \end{aligned}$$

Thus,

$$\begin{aligned}\textbf{wt }(w) &= [S(w,v)] \\ &= (1+r(2n-2)!)\mathrm{wt}\ (v) \\ &= (1+r(2n-2)!)[S(v,w)].\end{aligned}$$

But since v is of odd length,

$$[S(w,v) = [S(w,v)^T] = [S(w,v)]^T,$$

by the Suslin identities..
But $S(v,w)S(w,v)^T = I$, and so $[S(v,w)] = [S(w,v)]^{-1}$.
Thus, one gets $(2+r(2n-2)!)\mathrm{wt}\ (v) = 0$. A contradiction except when $n=2, r=-1$.

THANK YOU