Branching random walks with displacements coming from a power law

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- This model was introduced by Hammerseley (1974), Kingman (1975) and Biggins (1976).

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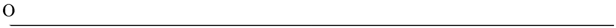

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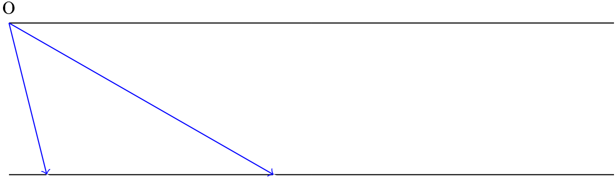

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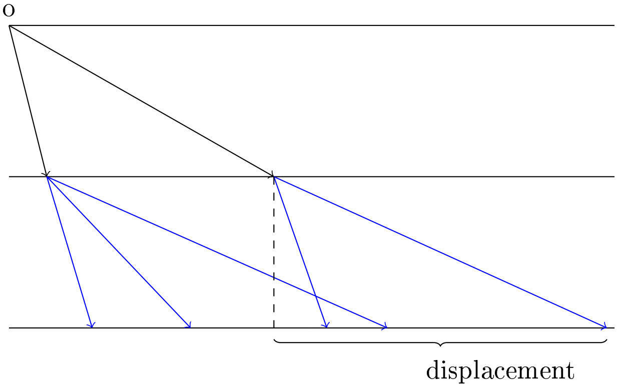

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- Each particle of Generation 1 undergoes the same random mechanism (i.e., branching and displacements) independently of each other (and of branching) to give rise to the Generation 2.
- This dynamics goes on.

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- In our rather simplified model, we only allow the particles to move along a single ray.
- More complicated models can also be considered where particles move in a plane or in a box.
- For the purpose of this talk, we shall restrict ourselves to the simple model and study long run configuration of the positions of particles.

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Conjecture (Brunet and Derrida, J. Stat. Phys. (2011))

- *The limiting configuration is a "Decorated Poisson Point Process".*
- *The limiting configuration exhibits "Property of Superposability".*

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- The last condition says that the displacements are heavy tailed in contrast to being Gaussian, which would have meant that $P(Disp > x) \sim (2\pi)^{-1/2} x^{-1} e^{-x^2/2} \ll cx^{-\alpha}$ for large x.

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- The above results confirm the Brunet-Derrida conjectures for branching random walks with heavy tailed displacements.

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- This new superposed point configuration has the same distribution as the original point configuration N_* (**Superposability property**).
- The point process N_* is also called α-stable point process; Davydov, Molchanov and Zuyev (2008, 2011)

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- In the context of rightmost particle, we have slightly improved an existing result of Durrett (1979, 1983).
- Most of the other results are not available in the light tailed case.

Thank You