Sharp Inequalities and related problems

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Inequalities : Discrete

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For $x_i, y_i \in \mathbb{R}, \; i = 1, ..., n$

$$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}}$$

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$$\frac{1}{(b-a)^2}\int_a^b |u|^2 dt \;\leq\; \int_a^b |u'|^2 dt$$

Poincaré Inequality

There exists an optimal constant $\lambda_1 > 0$ such that

$$\lambda_1 \int_a^b |u|^2 dt \leq \int_a^b |u'|^2 dt$$

holds for all functions $u \in C^1([a,b]), u(a) = u(b) = 0.$

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Let Ω be a bounded domain in $\mathbb{R}^n$, then there exists an oplimal constant $\lambda_1(\Omega) > 0$ such that

$$\lambda_1(\Omega) \int_\Omega |u(x)|^2 \, dx \;\leq\; \int_\Omega |\nabla u(x)|^2 \, dx$$

holds for all $u \in C_c^1(\Omega)$.

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$\lambda_1(\Omega)$ is the first eigen value of $-\Delta$

Inequalities in $\mathbb{R}^n$

Can we have this inequality in $\mathbb{R}^n$?:

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More generally can we have

$$C \left[\int_{\mathbb{R}^n} |u(x)|^q \, dx \right]^{\frac{p}{q}} \;\le\; \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \;, \quad u \in C_c^1(\mathbb{R}^n)$$

Inequalities in $\mathbb{R}^n$

Suppose we have such an inequality for some p, q then

$$q \;=\; \frac{np}{n-p} \;:= p^*$$

Sobolev Inequality

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$$S_{p,n} \left[\int_{\mathbb{R}^n} |u(x)|^{p^*} \, dx \right]^{\frac{p}{p^*}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \ , \quad u \in C_c^1(\mathbb{R}^n)$$

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Morrey's Inequality : Let $p > n$, there exists an optimal constant $S_{p,n} > 0$ such that

$$\sup_{x \neq y} \frac{u(x) - u(y)}{|x - y|^{1 - \frac{n}{p}}} \leq S_{n,p} \left[\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \right]^{\frac{1}{p}}$$

Best Constants and Extremals

First consider the case $p = 1$. i.e we have the inequality

$$S_{1,n}\left[\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx\right]^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|\, dx\ , \quad u \in C_c^1(\mathbb{R}^n)$$

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$$S_{1,n} = n^{1-\frac{1}{n}} \left[\omega_{n-1} \right]^{\frac{1}{n}}$$

where $\omega_{n-1} = H^{n-1}(\mathbb{S}^{n-1})$, the surface measure of the boundary of unit ball in $\mathbb{R}^n$.

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$S_{1,n}$ is not achieved

Isoperimetric Inequality

Isoperimetric problem :Determine the shape of the closed plane curve having a given length and enclosing the maximum area

Isoperimetric Inequality

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary then

$$H^{n-1}(\partial\Omega) \;\geq\; n^{1-\frac{1}{n}}\,[\omega_{n-1}]^{\frac{1}{n}}\,|Vol\;\Omega|^{\frac{n}{n-1}}$$

Iquality holds iff Ω is a ball

Equivalence

Federer-Fleming : Sobolev inequality with $p = 1$ is equivalent to the Isoperimetric Inequality

Cartan-Hadamard Manifold

Conjecture : Let (M, g) be a Cartan-Hadamard manifold then the isoperimetric inequality :

$$H^{n-1}(\partial\Omega) \geq n^{1-\frac{1}{n}} [\omega_{n-1}]^{\frac{1}{n}} |Vol\ \Omega|^{\frac{n}{n-1}}$$

holds in (M, g).

- n=2, Weil, 1926
- n=3, Kleiner, 1992
- n=4, Croke,1984
- $n \geq 5$, Joel Spruck,et al...2019 (???)

The case $p = 2$

$$S_{2,n}\left[\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}}\, dx\right]^{\frac{n-2}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx\ , \quad u \in C_c^1(\mathbb{R}^n)$$

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Extremal functions exist

If u is an Extremal then u solves

$$-\Delta u = u^{\frac{n+2}{n-2}} \; , \; u > 0 \; , \; \int_{\mathbb{R}^n} |\nabla u|^2 \; < \; \infty.$$

Extremals

- Rotational Symmetry

Extremals

- Rotational Symmetry
- Conformal invariance

Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be a conformal map and u is a solution of the PDE, the

$$\tilde{u} := J(K)^{\frac{n-2}{2}} u(K(x))$$

is also a solution

Extremals

- All solutions are of the from

$$U(x) = \left[\frac{\sqrt{N(N-2)}\varepsilon}{\varepsilon^2 + |x - x_0|^2} \right]^{\frac{N-2}{2}}$$

for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$

Equivalent Geometric problem

Obata : Let g be a metric on the unit sphere $\mathbb{S}^n$ conformal to the standard metric on S^n, then g is of constant scalar curvature 1 iff g is of constant sectional curvature 1.

Yamabe Problem

Let (M, g) be a compact Riemannian manifold of dimension N and scalar curvature K_g, can we find a metric $\tilde{g}$ conformal to g such that $\tilde{g}$ has constant scalar curvature.?
If $\tilde{g} = u^{\frac{4}{N-2}} g$, then this is equivalent to solving the PDE

$$-\frac{4(N-1)}{N-2}\Delta_g u + K_g u = k u^{\frac{N+2}{N-2}}$$

where k is a constant.

Hardy-Sobolev -Maz'ya Inequality.

Let $2 \leq k < N$, $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^h$. Denote a point $x \in \mathbb{R}^N$ by $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^h$, then for $t \in [0, 2)$ there exists an optimal constant $S = S_{t,N,k} > 0$ such that

$$S \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(t)}}{|y|^t} dx \right)^{\frac{2}{2^*(t)}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} dx$$

holds for all $u \in D^{1,2}(\mathbb{R}^N)$ where $2^*(t) = \frac{2(N-t)}{N-2}$ and $0 \leq \lambda \leq \frac{(k-2)^2}{4}$.

Euler Lagrange Equation.

$$-\Delta u \; - \; \lambda \frac{u}{|y|^2} \; = \frac{u^{p(t)-1}}{|y|^t}, \;\; u > 0, \; u \in D^{1,2}(\mathbb{R}^N)$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^h = \mathbb{R}^N$

Classification of Solution

Question : Uniqueness of Solution

Difficulty : Lack of rotational symmetry

Classification of Solution.

- **(G Mancini , Sandeep)** The problem has Hyperbolic symmetry and showed that the Solution space is of $N - k + 1$ dimensional

Classification of solutions when $t = 1$.

Theorem

(Fabri, Mancini, S) Let u_0 *be the function given by*

$$u_0(x) = u_0(y,z) = c_{N,k}\left((1+|y|)^2 + |z|^2\right)^{-\frac{N-2}{2}}$$

where $c_{N,k} = \{(N-2)(k-1)\}^{\frac{N-2}{2}}$. *Then* u *is a solution of*

$$-\Delta u = \frac{u^{\frac{N}{N-2}}}{|y|} \text{ in } \mathbb{R}^N, u > 0, u \in D^{1,2}(\mathbb{R}^N)$$

if and only if $u(y,z) = \lambda^{\frac{N-2}{2}} u_0(\lambda y, \lambda z + z_0)$ *for some* $\lambda > 0$ *and* $z_0 \in \mathbb{R}^{N-k}$

Thank You