ECG and evolution of open quantum systems

S. Chaturvedi

Department of Physics
IISER Bhopal, Bhopal

Symposium on the Life and Work of
Prof. ECG Sudarshan

IAS Annual Meeting
University of Hyderabad

8-10 November 2019

Some justification for being here

I have neither been a student nor a post-doc with ECG nor have I written any papers with him. However, my doctoral thesis work on the 'Poisson representation technique for stochastic description of chemical kinetics' was largely inspired by his 'Diagonal repesentation'. At that stage in my career I had not met ECG but my supervisor, Crispin Gardiner had. He was a post doc at Syracuse University after completing his doctorate at Oxford with Robert Dalitz. I recall having asked him once : among the many great physicists that you have met who would you regard as the most impressive. Without batting an eyelid he said : ECG Sudarshan, for when I met him for the first time, I had a strange feeling that I am in the presence of someone who knows everything.

Many years later, I joined the Institute of Mathematical Sciences when ECG was its director. It was there that I realized how accurately Crispin had described the feeling one got when ECG was around.
Recently, purely by accident, I found a photograph taken almost twenty years ago at a meeting in the SNBNCBS. It is the only photograph in which I had the good fortune of appearing with ECG and also with CKM whom I have greatly admired for his no-handwaving approach to physics.

State of a physical system in Quantum Mechanics

- Hilbert space $\mathcal{H}$:

$$\text{Elements of } \mathcal{H} \ : \ |\psi\rangle, |\phi\rangle \cdots$$
$$\text{Scalar product : } \langle\psi|\phi\rangle$$

- Unit Ball $\mathcal{B} \in \mathcal{H}$:

$$\mathcal{B} = \{|\psi\rangle \in \mathcal{H} \mid \langle\psi|\psi\rangle = 1\}$$

- Physical states of a quantum system : Each trace one positive operator on $\mathcal{H}$ describes a possible physical state of a quantum system and vice versa

$$\text{States} \leftrightarrow \rho, \quad \rho \geq 0, \quad \text{Tr}\rho = 1.$$

(ρ : a positive operator on $\mathcal{H} \Leftrightarrow \langle\psi|\rho|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$)

Physical states can be divided into two categories

$$\rho \quad \begin{matrix} \nearrow & \text{Pure if } \rho^2 = \rho \\ \searrow & \text{mixed if } \rho^2 \neq \rho \end{matrix}$$

$$\rho \ \text{pure} \quad \Leftrightarrow \rho = |\psi\rangle\langle\psi| \quad |\psi\rangle \in \mathcal{B}$$

$$\rho \ \text{mixed} \Leftrightarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad |\psi_i\rangle \in \mathcal{B}, p_i \geq 0, \sum_i p_i = 1$$

Mixed states may be viewed as an ensemble of pure states $\{|\psi_i\rangle\langle\psi_i|\}$ weighted by a probability distribution $\{p_i\}$

Evolution

Given the initial state ρ_0 of a **closed** quantum system at time $t = 0$ its state at later times is governed by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \rho = [H, \rho], \;\; H : \text{Hamiltonian}$$

which can be formally integrated to yield

$$\rho(t) = U(t)\rho_0 U^\dagger(t), \quad U(t) = e^{-iHt/\hbar}, \quad U(t) \text{ unitary}$$

Schrödinger evolution can be viewed as a map taking density operators to density operators

$$\rho' = \Lambda\rho = U\rho U^\dagger;$$

Its unitary nature has the consequence that:

$$\rho \text{ pure} \Rightarrow \rho' \text{ pure} \;\; ; \;\; \rho \text{ mixed} \Rightarrow \rho' \text{ mixed}$$

ECG: Generalization of the Schrödinger evolution

What is the structure of the most general evolution which takes density operators to density operators ?

In a paper in 1961

E. C. G. Sudarshan, P. M. Mathews and J. Rau, Stochastic dynamics of quantum-mechanical systems, Phys. Rev. 121, 920 (1961)

Sudarshan Mathews and Rau posed and answered this question :

$$\rho' = \sum_\alpha W_\alpha \rho W_\alpha^\dagger; \qquad \text{(preservation of positivity of } \rho\text{)}$$

$$\sum_\alpha W_\alpha^\dagger W_\alpha = \mathbb{I} \qquad \text{(preservation of trace of } \rho\text{)}$$

Remarks

- Schrödinger evolution arises as a special case when there is only a single term in the sum.
- Though the SMR derivation contained a gap, the result in itself has had far reaching consequences. It yields the structure of evolution of **open** quantum systems - dynamics of a system S regarded as a subsystem of a larger system $S + R$ and provides a framework for discussing relaxation phenomena in quantum systems.
- The SMR evolution viewed as a map taking density operatrors to density operators is commonly referred to as the Kraus representation and the operators W_α therein as Kraus operators though SMR work preceded that of Kraus by almost ten years.

Gorini-Kossakowski-Sudarshan-Lindblad Master equation

The SMR evolution may be viewed as a generalisation of the integrated form of the Schrödinger equation. A natural question to ask is what is the generalisation of the Schrödinger equation implied by the SMR evolution. Under some specific assumptions this question was posed and answered in a paper by Gorini Kossakowski and Sudarshan

V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N -level systems, J. Math. Phys. 17, 821 (1976).

and independently by Lindblad in

G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. 48, 119 (1976).

The differential form of the SMR evolution (under certain assumptions) turns out to be

$$i\hbar \frac{\partial}{\partial t}\rho = [H, \rho] + \frac{1}{2}\sum_{kl} c_{kl}[F_k, \rho F_l^\dagger] + [F_k\rho, F_l^\dagger]$$

where F_k are operators on the Hilbert space of the system and c_{kl} are elements of a complex positive matrix.

This equation may be equivalently recast as

$$i\hbar \frac{\partial}{\partial t}\rho = [H, \rho] + \frac{1}{2}\sum_{j}[2V_j\rho V_j^\dagger - V_j^\dagger V_j\rho - \rho V_j^\dagger V_j]$$

popularly known as the Lindblad form of the master equation.

ECG and CPM

Open quantum systems necssarily involve considering a system composed of two subsystems : the subsystem S, the system of interest, and the subsystem R, the reservoir or the rest.

In quantum mechanics the Hilbert space associated with a composite system A+B is taken to be the tensor product of the respective Hilbert Spaces:

$$\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B$$

The tensor product structure of the Hilbert space describing composite systems brings in a new feature : entanglement

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \quad \begin{array}{l} \nearrow \quad \text{separable if } |\Psi\rangle = |\psi\rangle \otimes |\phi\rangle, \\ \\ \searrow \quad \text{nonseparable or entangled otherwise} \end{array}$$

This feature of quantum mechanics which has no classical counterpart occupies a privileged position in the field of Quantum Information Theory where one seeks to exploit such features of quantum mechanics to carry out tasks which are either difficult or impossible in the classical context.

In this field, because of its very nature and objectives , one is required to spell out what are the most general operations that one can perform on a quantum system. Naturally any such operation, referred to as a quantum channel, must take a state to a state i.e. map positive operators to positive operators preserving the trace. It turns out that all such positive maps can be divided into two categories:

$$\text{PM} \begin{array}{l} \nearrow \quad \text{CPM}, \\ \\ \searrow \quad \text{NCPM} \end{array}$$

In a finite dimensional Hilbert space $\mathcal{H}$, of dimension d, in view of the natural correspondence between operators on $\mathcal{H}$ and $d \times d$ complex matrices, one can view this categorization as follows:

- PM are those maps Λ which take a $d \times d$ positive matrix to a $d \times d$ positive matrix
- CPM are PM and so are their trivial extensions $\Lambda \otimes \mathbb{I}_n$ for all n
- NCPM are PM but their extensions cease to have this property.

Sudarshan's 1961 work in the context of quantum evolution gives a complete characterization of CPM.
An example of a map that belongs to the NCPM category and has played an important role in detecting entanglement is the transpose map:

$$\Lambda(\rho) = \rho^T$$

The two papers of ECG cited earlier have been instrumental in initiating activity in a wide variety of areas in physics and mathematics:

1. Open Quantum Systems
2. Positive Maps
3. Quantum Markov processes
4. Dynamical Semigroups
5. ....