From Calculus to Number Theory

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$$\text{Integers} = \mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

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This was first proved by Nicole Oresme (1323-1382), a brilliant French philosopher of his times.

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This went by the name "The Basel Problem", posed first by Pietro Mengoli in 1644, and was unsuccessfully–but famously–attacked by the Bernoulli family.

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Suggested reading: *A journey through genius: great theorems of Mathematics*, by William Dunham, published by Penguin.

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Proof: Put $x = 1$ in the expansion

$$\tan^{-1}(x) \; = \; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

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are prototypical examples of an exciting area of research in modern number theory that goes by the appellation:

Special Values of L-functions.

The Riemann zeta function

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Euler's formula may be stated as $\zeta(2) = \frac{\pi^2}{6}$. It was proved by Apery in 1979 that $\zeta(3)$ is irrational. More generally, $\zeta(2m)$ is very well-understood but $\zeta(2m+1)$ is a total mystery.

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Madhava's formula may be stated as $L(1,\chi) = \frac{\pi}{4}$ if χ is the unique nontrivial character modulo $N = 4$.

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Examples: Dirichlet's theorem on infinitude of primes in AP boils down to proving $L(1, \chi) \neq 0$. The value at $s = 1$ of the Dedekind zeta function of a number field has information about important invariants about that number field–the class number formula!

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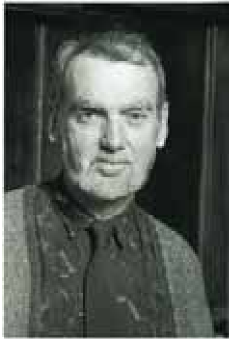

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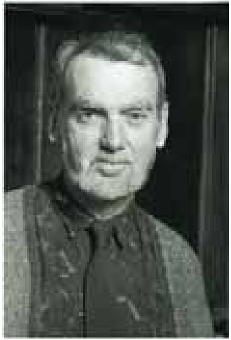

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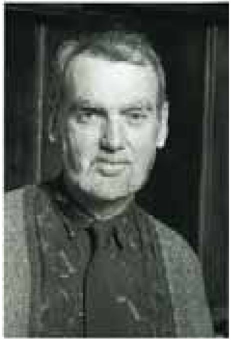

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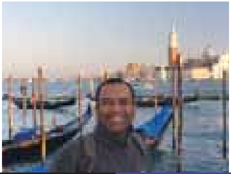

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