Towards a Convex-Analytic View of Impossibility Results in Stochastic Control and Information Theory

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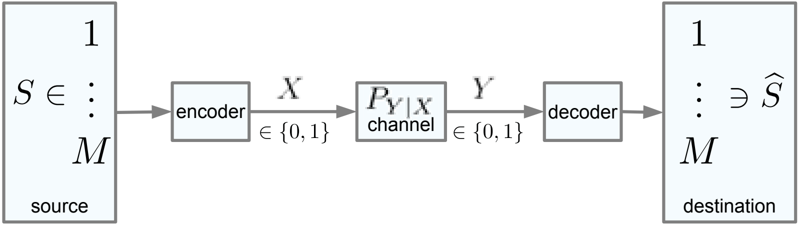
November 5, 2016

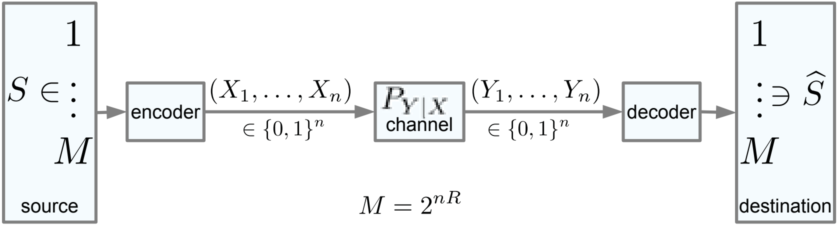
Shannon's noisy channel coding theorem (1944)

- Standard communication system



- Naive approach: naively encode and send each message multiple times, average out to decode
- Shannon's approach: block coding, encoding with redundancy and decoding so that chance of error becomes small

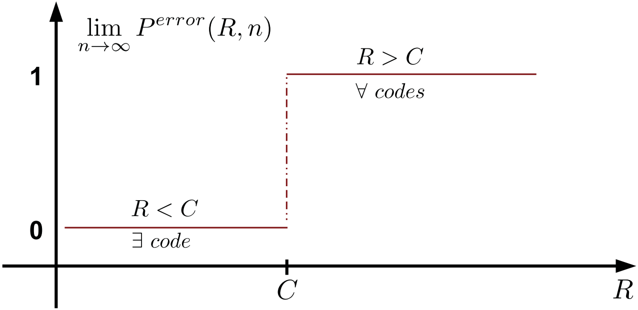
Shannon's noisy channel coding theorem (1944)



- Shannon considers the *block-coding* setup
- Shannon's channel coding theorem

$$P^{error}(code, R, n) = Prob(S \neq \widehat{S})$$

$$P^{error}(R, n) = \min_{codes} \; P^{error}(code, R, n)$$



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- Channel capacity

$$C = \max_{p_X} I(X;Y)$$

$$I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

- Tremendous drop in complexity; “single letter characterization”

$$(2^n)^{2^{nR}} \times (2^{nR})^{2^n} \;\; codes$$

Open questions

- **Question:** What about finite blocklength performance?
- Discrete optimization. Extremely hard to compute as n gets large

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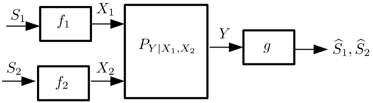

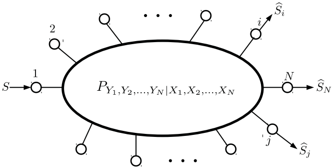

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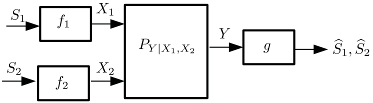

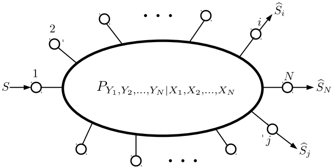

- Shannon invented a "calculus" of mutual information to prove theorems of information theory
- This calculus works well for information theoretic problems and fails miserably for other problems
- **Question:** Proofs and generalizations

Finite blocklength performance bounds

- Focus on lower bounds
- **Existing approach:** concentration inequalities, Gaussian approximations

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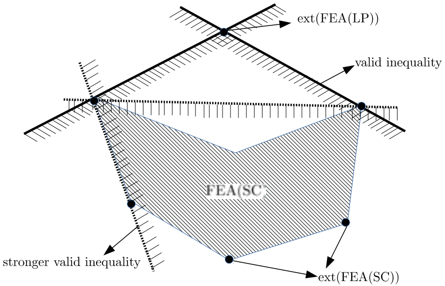
Our approach

Convex analytic geometry of the finite blocklength problem.



[Jose and Kulkarni, 2015, IEEE CDC], [Jose and Kulkarni, 2016, IEEE Trans IT]

Original problem

$$\min_{codes} \; P^{error}(code, R, n)$$

$\Longleftrightarrow$

Optimization over distributions

(Nonconvex!)

$$\min_{distributions} \; P^{error}(distribution, R, n)$$

$\Downarrow$

Convex relaxation

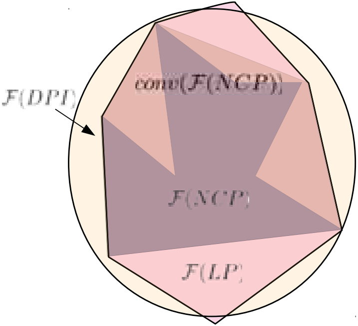
$$\min_{convex\ relaxation} \; P^{error}(dist, R, n)$$

$\Longleftarrow$

A particular LP relaxation

$$\min_{hyperplanes} \; P^{error}(dist, R, n)$$

Power of the LP relaxation



1. One can **systematically obtain** lower bounds

Infotheory problem	$\geq$	**value of LP relaxation**	$=$	**value of dual LP**	$\geq$	**value of any dual feasible point**

2. Gives a suite of lower bounds
3. Customizable, generalizable

Main results: point to point

Theorem (point to point)

1. *Asymptotically tight finite blocklength lower bounds for*
   - *Channel coding*
   - *Source coding*
   - *Joint source-channel coding*
2. *Improvement over previous finite blocklength bounds*

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Theorem

Consider a discrete memoryless binary symmetric channel with cross over prob ϵ*. For any code,*

$$Prob\{S \neq \widehat{S}\} \geq$$

$$\sup_{\epsilon > \delta > 0} \Big[1 + 2^{-n\left(H_2(\epsilon) - H_2(\epsilon-\delta) - \delta \log_2 \frac{1-\epsilon}{\epsilon} + \frac{1}{n} \log_2 l(n, \epsilon-\delta)\right)} - \frac{1}{M} 2^{n\left(1 - H_2(\epsilon) + \delta \log_2 \frac{1-\epsilon}{\epsilon}\right)}$$

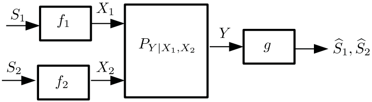
$$- (1-\epsilon)^{n - nH_{\frac{1}{1-\epsilon}}(\epsilon-\delta)} \Big],$$

$$\textit{where} \quad l(n, \alpha) = \frac{\exp(\lambda_1(n) - \lambda_2(n\alpha) - \lambda_2(n(1-\alpha)))}{\sqrt{2\pi\alpha(1-\alpha)n}}$$

$$\textit{with} \quad \lambda_1(x) = \frac{1}{12x+1}, \quad \lambda_2(x) = \frac{1}{12x}, \quad \textit{and}$$

$$H_q(x) = x \log_q(q-1) - x \log_q(x) - (1-x) \log_q(1-x).$$

Main results: networks



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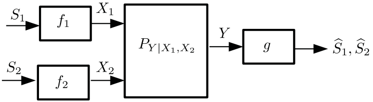

Theorem (Converse for multiple access channel)

For any code,

$$\mathbb{P}[(S_1, S_2) \neq (\widehat{S}_1, \widehat{S}_2)] \geq \text{OPT(MAC)} \geq \text{OPT(DPMAC)}$$

$$\geq \sup_{\gamma>0, U, \bar{Y}} \max \Bigg(\sup_{a_1, a_2 : a_1 > a_2 \geq 0} \inf_{x_1, x_2} \mathbb{P}[a_1 i(x_1; \bar{Y}|x_2, U) + a_2 i(x_2; \bar{Y}|U) \leq a_1 \log M_1 + a_2 \log M_2 - \gamma] - \frac{\exp(-\gamma/a_1) \mid \mathcal{U} \mid}{\inf_{y, x_2, u : P_{\bar{Y}|X_2,U}(y|x_2,u)>0} P_{\bar{Y}|X_2,U}(y|x_2, u)^{(a_2/a_1)}},$$

$$\sup_{a_1, a_2 : a_2 > a_1 \geq 0} \inf_{x_1, x_2} \mathbb{P}[a_1 i(x_1; \bar{Y}|U) + a_2 i(x_2; \bar{Y}|x_1, U) \leq a_1 \log M_1 + a_2 \log M_2 - \gamma] - \frac{\exp(-\gamma/a_2) \mid \mathcal{U} \mid}{\inf_{y, x_1, u : P_{\bar{Y}|X_1,U}(y|x_1,u)>0} P_{\bar{Y}|X_1,U}(y|x_1, u)^{(a_1/a_2)}},$$

$$\sup_{a_1 : a_1 = a_2 > 0} \mathbb{P}[a_1 i(x_1, x_2; \bar{Y}|U) \leq a_1 \log M_1 M_2 - \gamma] - \mid \mathcal{U} \mid \exp(-\gamma/a_1) \Bigg),$$

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- Resolve open issues about role of Shannon theory in stochastic decentralized control [**Kulkarni and Coleman, 2015, IEEE TAC**]
- Nearly resolved a 60 year old open combinatorial problem posed by Levenshtein for the deletion channel [**Kulkarni and Kiyavash, 2013, IEEE Trans IT**], [**Kulkarni et. al, 2013, Discrete Mathematics**]

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Looking forward

- Convex analytic theory of fundamentally probabilistic results
- "Geometry of probability"

Acknowledgments

- DST
- IRCC, IIT Bombay

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Thank you!