Quantum isometry groups

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Outline

Classical	Quantum	Background
Compact Hausdorff space	Unital C^* algebra	Gelfand-Naimark
Compact Group	Compact Quantum Group	Woronowicz
Group Action	Coaction	Woronowicz
Riemannian manifold	Spectral triple	Connes
Isometry group	Quantum Isometry Group	To be discussed

Alain Connes' question (1995)

What is the quantum symmetry group of a NONcommutative space?

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Answer

1. In 1998 Wang defined the notion of quantum symmetry groups for finite NONcommutative spaces.
2. Banica and Bichon defined quantum symmetry groups for finite metric spaces, finite graphs, etc.
3. Lots of examples computed leading to discovery of completely new kinds of quantum groups

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Remark

The above examples dealt with some finite or discrete structures.

Quantum isometry groups (Bhowmick + Goswami, J.F.A., 2009)

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Ingrediants

1. Geometry given by Spectral triples on noncommutative C^* algebras.
2. Symmetry given by Compact Quantum Groups (CQG) and their coactions.
3. Isometric coactions

Results for classical spaces

Theorem(Bhowmick + Goswami, Journal of Functional Analysis, 2009)

$QISO^{+}(C^{\infty}(M), L^{2}(S), D)$ is classical for $M = S^{1}, \ \mathbb{T}^{2}, S^{n}$.

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Theorem (Bhowmick + Banica, De Kommer, Ann. Math. Blaise Pascal, 2012)

A classical Riemannian manifold cannot have genuine group dual isometries. In particular, all the known examples of orthogonal easy quantum groups cannot coact isometrically on a classical Riemannian manifold.

For a group G, we compute $QISO^+(C_r^*(G))$.

Proposition(Bhowmick-Skalski, Journal of Geometry and Physics, 2010)

1. For $G = \mathbb{Z}_n, n \neq 4$, $QISO^+$ is commutative,$QISO^+ = C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$.
2. For $G = \mathbb{Z}_4$, $QISO^+ = C^*(D_\infty \times \mathbb{Z}_2)$. (noncommutative and infinite dimensional)

Proposition(Bhowmick-Skalski, Journal of Geometry and Physics, 2010)

For $G = \mathbb{Z}$, $QISO^{+}(C^{*}(\mathbb{Z})) = C(S^{1} \rtimes \mathbb{Z}_{2})$.

Generating set: $\{(1,2),(2,3)\}$.

Proposition(Bhowmick-Skalski, Journal of Geometry and Physics, 2010)

$QISO^+(C^*(S_3)) = C^*(S_3) \oplus C^*(S_3)$.

Remark, Dalecki-Soltan

This was extended to all S_n by Dalecki-Soltan.

Theorem (Bhowmick-Skalski, Journal of Geometry and Physics, 2010)

$QISO^+(C_r^*(\mathbb{F}_2)) \cong H_{2,0}^+$ of Banica-Skalski.

New 2 parameter family of CQG (Banica-Skalski)

$H_{p,q}^+$ is the universal C^* algebra generated by

1. $(2p+q)^2$ partial isometries $\{U_{z,y} : z, y \in \tau_{p,q}\}$
2. For $i\alpha, j\beta \in \tau_{p,q},\ M, N \in \{1, 2, ..., q\}$,
3. $\{U_{z,y} : z, y \in \tau_{p,q}\}$ forms a unitary matrix.
4. $U_{i\alpha,N}^* = U_{\bar{i}\alpha,N}$
5. $U_{M,j\beta}^* = U_{M,\bar{j}\beta}$
6. $U_{M,N}^* = U_{M,N}$

Moreover, $QISO^+(C_r^*(\mathbb{F}_p)) \cong H_{p,0}^+$.

Computation for other noncommutative C^* algebras

Theorem (Bhowmick + Goswami, Journal of Functional Analysis, 2009)

$QISO^+$ for the Chakraborty-Pal triple on $SU_\mu(2)$ is $U_\mu(2)$.

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Theorem (Bhowmick + D. Goswami, A. Skalski, *Trans. Amer. Math. Soc.*, 2011)

Quantum isometry group commutes with inductive limits.

Set up

1. A C^* dynamical system $(A, \mathbb{T}^n, \beta)$
2. A faithful representation $\pi_0 : A \to B(H)$ and a spectral triple (A^∞, H, D, R).
3. A compact abelian group $\widetilde{\mathbb{T}^n}$ with a covering map $\gamma : \widetilde{\mathbb{T}^n} \to \mathbb{T}^n$, Lie algebras of both $\widetilde{\mathbb{T}^n}$ and $\mathbb{T}^n$ identified with $\mathbb{R}^n$.
4. A strongly continuous unitary representation V of $\widetilde{\mathbb{T}^n}$ on H such that $V_{\tilde{g}} D = D V_{\tilde{g}}$ and $V_{\tilde{g}} \pi_0(a) V_{\tilde{g}}^{-1} = \pi_0(\beta_{\gamma(\tilde{g})}(a))$

Theorem (Bhowmick + Goswami, Journal of Functional Analysis, 2009)

1. (A_J^∞, H, D) is also a deformed spectral triple,
2. $QISO^+(A_J^\infty, H, D) \cong [QISO^+(A^\infty, H, D)]_{J \oplus -J}$.