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Algebraic geometry in India

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1. Introduction

This article will be mainly concerned with the development of algebraic geometry in India during the second half of the twentieth century.

Algebraic geometry is a central field of mathematics with intimate connections with other major areas like number theory and complex analysis, and also with mathematical physics. During the period under consideration the subject has been revolutionised by the introduction of new concepts and techniques by Grothendieck and others; this progress has been instrumental in solving outstanding and famous problems not only in algebraic geometry but also in related fields like number theory.

Mathematicians from India have made influential and extensive contributions to algebraic geometry, by way of initiating new areas, providing important breakthroughs and developing useful techniques. In order to give an idea of the scope and depth of the contributions from India, I will describe mainly some work in four topics with which I am familiar: Moduli problem of vector bundles (and the related geometric invariant theory), the work of C P Ramanujam, Frobenius split varieties and algebraic cycles.

Indian mathematicians were introduced in the 1950s to modern algebraic geometry *via* the lectures at TIFR by Laurent Schwartz on complex analytic manifolds, and through the works of Kodaira, Spencer and Serre.

2. Vector bundles and moduli

In response to a question of Serre, as to whether every vector bundle on the affine space is trivial (equivalently, whether every projective module over a polynomial ring over a field is free), C S Seshadri proved in 1958 that vector bundles on the affine plane are trivial. The general case was settled by D Quillen and A A Suslin about 15 years later, not long after M P Murthy and J Towber settled the three-dimensional case.

In 1965, M S Narasimhan and Seshadri proved the fundamental result relating stable vector bundles on a compact Riemann surface to unitary representations of the (orbifold) fundamental group of the surface. This celebrated and influential result has served as a model for a large amount of literature connecting algebraic geometry, differential geometry and topology and was generalised in various directions. Seshadri (who had earlier constructed the Picard variety of a complete variety and proved its universal property) also constructed the moduli spaces of vector bundles on curves. He also introduced and studied the notion of *parabolic bundles*.

Starting from late sixties, Narasimhan and S Ramanan undertook an extensive and deep study of moduli space of vector bundles on curves. This work (which also revealed surprising connections between classical algebraic geometry and moduli of vector bundles on curves) attracted much attention from mathematicians and physicists. They introduced the notion of Hecke correspondence between moduli spaces of vector bundles and this has

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become a basic tool in the study of these spaces. Using the Hecke correspondence, they constructed an explicit desingularisation of the moduli space of vector bundles of rank 2 with trivial determinant and proved that the number of deformation parameters of the moduli space (with fixed determinant) is the same as that of the curve, when the rank and degree of the bundle are coprime.

G Harder and Narasimhan used the Weil conjectures to compute the Betti numbers of these spaces. In this work they introduced what is now known as the *Harder–Narasimhan filtration*. This filtration and its analogues have proved to be immensely useful in several contexts of algebraic and arithmetic geometry, and more recently in commutative algebra.

Seshadri worked also on geometric invariant theory, originally motivated by its connection with moduli problems. Using global geometric techniques, he proved Mumford's conjecture (that a linear representation of a reductive group is geometrically reductive) in the case 'stable = semistable'. In this work he also proved a very useful criterion for ampleness of a line bundle, now known as Seshadri's Criterion, leading to the recent literature on 'Seshadri constants'. The Mumford conjecture was proved by W Haboush; recently Seshadri has given a geometric proof of this theorem.

Ramanan and A Ramanathan made an incisive study of the instability flags arising in geometric invariant theory. They used this to give an algebraic proof of the semistability of bundles obtained from semistable bundles by extension of structure groups in characteristic zero. If the characteristic of the field is positive, their method enables one to analyse the semistability of the extended bundle.

The general theme of vector bundles on curves plays an important role in the work of L Lafforgue on the Langlands conjecture for function fields and the work of G Laumon and B C Ngo on the 'fundamental lemma'.

The pioneering work at TIFR on moduli theory was followed up in many directions by a number of younger mathematicians. Ramanathan initiated the study of principal bundles on curves. In a series of works motivated by theoretical physics, J Drezet, T R Ramadas, Ramanathan and S Kumar (together with Narasimhan) made an extensive study of line bundles and linear systems on these moduli spaces. Bhosle studied moduli spaces of bundles on hyperelliptic curves, and curves with singularities. N Nitsure constructed the moduli space of Higgs bundles on curves as an algebraic variety and proved that the Hitchin map is proper; he also studied the moduli spaces of D-modules.

In another direction, V B Mehta and Ramanathan proved the very useful restriction theorem for semistable bundles.

3. Work of C P Ramanujam

An important result of Ramanujam, obtained in 1972, is his vanishing theorem: *If X is smooth projective surface (over $\mathbb{C}$) and L a line bundle on X, which is nef and big; then $H^i(X, L^{-1}) = 0$ for $i \neq 2$. ('nef' means $(L \cdot C)$ is non-negative for all curves contained in X, and 'big' means that the self-intersection number $(L \cdot L)$ is strictly positive).*

This result was generalised to higher dimensions by Y Kawamata and E Viehweg and plays a very important role in the geometry of higher dimensional varieties, where it is one of the important tools used in the minimal model programme.

Another famous theorem of Ramanujam gives a topological characterisation of the affine plane as an algebraic variety: *Let X be a smooth complex affine surface which is contractible and is simply connected at infinity; then X is isomorphic to the complex plane as an algebraic variety.*

4. Frobenius splitting

Let X be a variety over an algebraically closed field of positive characteristic and F the Frobenius endomorphism of X. If the canonical map of sheaves

$$\mathcal{O}_X \to F_* \mathcal{O}_X$$

is a split inclusion, we say that X is Frobenius split. This notion was introduced by Mehta and Ramanathan in 1985. The reason for its importance is that if X is Frobenius split and L is an ample bundle on X, then all higher cohomologies of L on X vanish. Mehta and Ramanathan proved that G/P where, G is semi-simple and P is a parabolic subgroup, is Frobenius split (as also all Schubert varieties in G/P) and used this to prove vanishing theorems for line bundles on flag varieties and results on linear systems on these varieties in positive characteristic. Ramanan and Ramanathan proved that Schubert varieties are arithmetically normal.

5. Abelian varieties

Ramanan studied the very ampleness of ample bundles on abelian surfaces with applications to the Horrocks–Mumford bundle. Narasimhan and Nori proved that there are only finitely many smooth projective curves having a given abelian variety as Jacobian by showing that the number of orbits of the automorphisms group of an abelian variety on the set of principal polarizations is finite.

M V Nori showed that the Griffiths group (of homologically trivial algebraic cycles modulo algebraic equivalence) of a generic Jacobian of dimension ≥ 3 is not finitely generated. His technique, of using Hecke operators to construct new cycles has been used by several others to give other examples of infinite generation.

In a different direction, Mehta and V Srinivas showed that in positive characteristics, an ordinary variety with a trivial tangent bundle has a finite degree covering space which is an ordinary abelian variety.

6. Algebraic cycles

There have been interesting Indian contributions to the theory of algebraic cycles. The result of Nori on Griffiths groups was mentioned earlier. Work of Murthy, N Mohan Kumar, and later S M Bhatwadekar, related affine geometry to algebraic cycles. K H Paranjape computed the Chow groups of general complete intersections of low multidegree. Srinivas and his collaborators have studied cycles on singular varieties in a systematic way.

7. Other topics

Here we briefly discuss some other topics, representing work in algebraic geometry which has not been considered in the previous sections.

One important series of works, by Seshadri in collaboration with V Lakshmibai, C Musili, and others, is the theory of standard monomials, which began with a goal of giving characteristic-free descriptions of homogeneous coordinate rings of homogeneous spaces for classical groups, and ultimately became an important technique in representation theory.

Several Indian mathematicians have also made fundamental contributions in the area of affine algebraic geometry, inspired by Ramanujam's theorem characterising the complex affine plane, on the one hand, and Seshadri's early work on the Serre problem, on the other. This includes the work of Murthy, Mohan Kumar, Nori, R Parimala, Bhatwadekar, R V Gurjar, A R Shastry and many others. Topics considered include the study of their topology, as well as their algebro-geometric aspects, like embedding questions, the complete intersection property, defining equations, and the related study of vector bundles (projective modules), possibly endowed with additional structures. Some of this overlaps with the theory of Lie and algebraic groups, which is discussed in another article. One influential result we mention here is Nori's proof of a vast generalisation of the Zariski conjecture regarding the fundamental group of the complement of a nodal curve in the projective plane.