Current Trends
in Science

platinum jubilee special

Lie groups and algebraic groups

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We give an exposition of certain topics in Lie groups and algebraic groups. This is not a complete survey and is meant to highlight certain aspects with which the authors are familiar, and to which there has been a significant contribution from mathematicians working in India.

1. Introduction

One of the fundamental branches of mathematics is the study of symmetries of any mathematical object (like a geometric object, or a differential/polynomial equation, etc.). The symmetries form a *group*, and group theoretical properties of this group give us information on other mathematical properties of the object under consideration.

For example, the solvability of the 'Galois group' of a polynomial equation is equivalent to the solvability of the equation by radicals. A similar statement can be made for solvability of differential equations.

The most ubiquitous class of groups is that of Lie groups. These are manifolds (geometrical objects, functions on which can be differentiated) such that the group operations are infinitely differentiable. Lie groups occur naturally in geometry, analysis, algebra and physics. Prototypical examples are the groups $GL_n(\mathbb{R})$ ($GL_n(\mathbb{C})$) of invertible $n \times n$ matrices with real (complex) entries.

By its very nature, a Lie group has analytic, geometric and algebraic properties, each of which is a highly developed field of study. These fields are interrelated and each of these fields contributes to the other.

2. Examples and classification

We first give some examples of Lie groups. The most frequently occurring ones are the linear classical groups $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SO_n(\mathbb{R})$, $SO_n(\mathbb{C})$, $Sp_n(\mathbb{R})$ and $Sp_n(\mathbb{C})$. Let us call a connected Lie group simple if G has no connected normal closed subgroups other than G and the identity. It is a remarkable fact that simple Lie groups can be completely classified; they are the special linear groups, orthogonal groups and symplectic groups. Apart from these, the list is a finite one (the so-called exceptional groups). This is the Cartan–Killing classification, which nowadays, is described in terms of the 'Dynkin diagrams'.

A related class of groups are simple algebraic groups. They can be defined over an arbitrary field. For a discussion on classification of these groups, see section 5.

3. Representation theory

3.1 *Finite dimensional representations*

Lie groups are essentially (up to coverings) closed subgroups of $GL_n(\mathbb{R})$. One of the fundamental problems in the theory of Lie groups was to find all possible ways in which a Lie group can be (embedded as) a subgroup of $GL_n(\mathbb{R})$ (or $GL_n(\mathbb{C})$). This is a problem which is completely solved for connected Lie groups.

The foregoing problem is equivalent to finding all possible group homomorphisms $G \to GL_n(\mathbb{C})$ – with varying n – which are infinitely differentiable

Keywords. Lie groups; representations; algebraic groups; arithmetic groups, rigidity.

(smooth). Such homomorphisms are called (finite dimensional) 'representations' of G. In other words, elements of G may be viewed as invertible linear transformations of the vector space $\mathbb{C}^n$. By an abuse of notation, the vector space $\mathbb{C}^n$ together with the G-action, is also called a representation of G.

We will describe – briefly – the solution of this problem, for *compact* connected Lie groups. A compact Lie group G may be shown (with some work, using the theory of compact operators) to be none other than a closed subgroup of the unitary group $U(n)$ of $n \times n$-matrices $g \in GL_n(\mathbb{C})$ with ${}^t\overline{g}g = 1$, for some n. If G is connected, it is possible to replace G by a conjugate in $U(n)$ such that the intersection with G of the group of diagonals in $U(n)$ is a maximal connected abelian subgroup T of G (it is called a 'maximal torus' in G). All maximal tori in G are conjugate (this purely algebraic statement was first proved by using the Lefschetz fixed point theorem, which is a fundamental result in algebraic topology; this is an instance of geometric properties of G yielding algebraic information).

Direct sums and tensor products of representations may be defined in an obvious way. A representation is said to be irreducible, if it cannot be written as a direct sum of two representations. If G is compact, every representation can be written as a direct sum of irreducible ones. Therefore, the problem of describing representations reduces to that of irreducible representations. It can be shown that if $\rho : G \to GL_n(\mathbb{C})$ is irreducible, the trace function $g \mapsto \mathrm{trace}(\rho(g))$ on G completely determines the representation.

To continue, one may show that the quotient manifold G/T is a compact complex manifold such that the group G (which acts on G/T by translations), acts by holomorphic automorphisms. If χ is a character of T (a continuous homomorphism of T into $U(1) =$ the circle group), then one may form a 'line bundle' $\mathcal{L}_\chi$ on G/T which can be shown to be a holomorphic line bundle. The space $H^0(G/T, \mathcal{L}_\chi)$ of holomorphic sections can be shown to be finite dimensional, on which G acts by left translations. One has the fundamental theorem of Hermann Weyl, Armand Borel and Andre Weil:

Theorem 1. *The representation* $\rho = \rho_\chi = H^0(G/T, \mathcal{L}_\chi)$ *is irreducible. Moreover, every irreducible representation of* G *may be realised in this way, for some character* χ *of* T. *The trace function* $\mathrm{trace}(\rho(g))$ *can be completely described in terms of* χ.

Historically, Hermann Weyl first described the irreducible representations in terms of their traces, without an explicit construction of the representations. The explicit realisation is due to Borel and Weil.

An important problem in physics is the decomposition of a representation ρ of a compact Lie group restricted to a compact subgroup H. In particular, the decomposition of a tensor product is especially important; the PRV conjecture (see [12]) describes exactly what happens in this case for certain important constituents of the tensor product; the conjecture was proved by Shrawan Kumar [25]. In a different direction, a unique factorisation theorem for tensor products was proved by C S Rajan [22].

3.2 *Infinite dimensional representations*

We will now consider unitary representations of G i.e. infinite dimensional Hilbert spaces H on which non-compact Lie groups G act preserving the inner product. A special class of Lie groups are those whose Lie algebra is a direct sum of irreducible (finite dimensional) representations of G each of which has dimension strictly greater than one. Such groups are called semi-simple Lie groups. It is no longer true that a representation is a direct 'sum' of irreducibles. The direct sum has to be replaced by a direct integral.

A basic example of a representation is the space $L^2(G)$ of the space of square integrable functions on G (square integrable with respect to the Haar measure). Even in this case, a description of the direct integral decomposition is a formidably difficult problem. It was however, completely solved by the great Indian mathematician Harish-Chandra in a series of papers.

The description of representations which occur, together with the number of times they occur (the multiplicity, or the Plancherel measure in the case of direct integrals) in the direct integral decomposition of $L^2(G)$ is called the Plancherel theorem. The classification of these representations contributing to the Plancherel theorem is in terms of fundamental building blocks, which are called 'discrete series representations'. Harish-Chandra proved that a semi-simple Lie group G has discrete series if and only if G has a connected maximal abelian subgroup T which is compact (a compact Cartan subgroup T). Harish-Chandra gave a formula (the Plancherel measure) for the representations, together with multiplicities, which occur in $L^2(G)$. The Plancherel measure for G/K (K a maximal compact subgroup of G) was computed for many classical groups by T S Bhanu-Murthy [4]. The complete classification of the representations occurring in $L^2(G)$ was achieved by Knapp and Zuckerman [5].

Somewhat similar to the case of compact Lie groups, Harish Chandra gave a formula – without explicitly constructing the discrete series representations – for the 'trace distribution' of discrete series. The problem of explicitly constructing the discrete series was solved by various people. Many Indian mathematicians contributed to this development: an algebraic construction was given by K R Parthasarathy, Ranga Rao and V S Varadarajan. A description, closer in spirit to the Borel–Weil (–Bott) theorem, was conjectured by Langlands, and proved by M S Narasimhan–Okamoto (in the case when the symmetric space of the group G is a Hermitian symmetric domain) and by R Parthasarathy (for G arbitrary but the weight of the discrete series is sufficiently regular). A complete proof of Langlands' conjecture was given by Wilfried Schmidt.

The quotient G/T has many different complex structures which are G-invariant. Fix one. Fix a character χ of T. One obtains a holomorphic line bundle $\mathcal{L}_\chi$ over G/T analogous to the compact situation. Consider the so-called 'Dolbeault complex' of square summable differential forms with coefficients in $\mathcal{L}_\chi$ on the complex manifold G/T, and for $p \geq 0$, let $\rho = \rho_{\chi,p}$ be the p-th cohomology group of this complex, considered as a G-module. Then Langlands' conjecture (Schmidt's theorem) is the following (compare with theorem 1).

Theorem 2. *For exactly one value of p, the representation ρ is an irreducible discrete series representation. Moreover, every discrete series representation arises this way.*

3.3 *Unitarizability*

The problem of describing all unitary irreducible representations of a semi-simple Lie group G is, as yet, not completely solved. For all classical groups, it is known by recent work of Vogan, Barbasch and others. The difficulty is in showing that certain irreducible representations of the Lie algebra have an (G-) invariant positive definite Hermitian form. This is the question of 'unitarizability' of representations of the Lie algebra and there has been a huge collaborative work recently on obtaining effective algorithms for deciding when the Lie algebra representations do have such a Hermitian form. R Parthasarathy and Siddharth Sahi have contributed significantly to this problem.

4. Geometry and groups

4.1 *The Erlangen programme of Klein*

Let us recall the fundamental theorem of projective geometry, which says that any set theoretic map of projective space which preserves its 'geometry' (i.e. incidence relations such as subspaces of a given vector space) is actually a linear isomorphism. The proof is a repeated use of the high transitivity of the action by the group of linear isomorphisms on the 'geometry' of the projective space. Similarly theorems of affine geometry translate into properties of the affine transformation group. Similar results hold for other geometries.

Felix Klein had the idea that extension of these results to other geometries can be deduced from group theoretical properties of certain transformation groups associated to these geometries. He sketched a programme (the Erlangen programme) wherein he conjectured that all of geometry may be 'reduced' to group theory. The programme was partly successful.

4.2 *Tits buildings*

Jacques Tits completely turned this situation around and proved a beautiful generalisation of the fundamental theorem of projective geometry. He associated a certain geometry to groups G over fields k, and showed that for groups of k-rank ≥ 2, maps between these 'Tits buildings' arose from group homomorphisms; so theorems for these groups (such as finding generators and relations) were deduced from geometric properties of the buildings. Here, the *k-rank* of an algebraic group G over a field k is the dimension of a maximal k-subgroup of G, which is k-isomorphic to a product of copies of GL_1. This important group-theoretic notion occurs repeatedly in the sequel.

Generalisations of symmetric spaces were considered by Bruhat and Tits, and they constructed p-adic symmetric spaces – the 'Bruhat–Tits buildings'. The Bruhat–Tits buildings have had applications in many fields: in the theory of automorphic forms, where local zeta functions of Shimura varieties are related to properties of an associated Bruhat–Tits buildings, for cohomology computations in the series of papers by Raghunathan and G Prasad, in questions on classification of bundles by Raghunathan and Ramanathan, and in the work of Garland and Raghunathan on loop spaces.

5. Discrete subgroups of Lie groups

5.1 *Rigidity*

We begin with some examples. A compact Riemann surface of genus $g \geq 2$ may be viewed as a discrete subgroup Γ of the Lie group $SL_2(\mathbb{R})$ of 2×2-real matrices with determinant one. The discrete subgroup will further have the property that the quotient $\Gamma\backslash G$ is compact. Moreover, the discrete subgroup is determined only up to conjugacy. The discrete subgroup is simply the fundamental group of the Riemann surface, which by the uniformisation theorem, may be thought of as the deck transformation group of the universal covering space (the upper half plane) of the Riemann surface. It can be shown that the space $\mathcal{M}_g$ of Riemann surfaces of genus g, is a complex algebraic space of dimension $3g-3$ i.e. of real dimension $6g-6$. However, all these surfaces have the same topology; hence the space of (non-conjugate) homomorphisms of Γ into $SL_2(\mathbb{R})$ (which are injective with discrete image) is a 'continuous space' of dimension $6g-6$. In particular, Γ may be continuously deformed in $SL_2(\mathbb{R})$ with the deformation space being $(6g-6)$-dimensional.

If $SL_2(\mathbb{R})$ is replaced by other (semi-) simple groups (e.g. $SL_n(\mathbb{R})$ with $n \geq 3$), then the space of deformations of a cocompact discrete group Γ may similarly be considered. However, repeated attempts of constructing such deformations failed, and the only known examples arose from arithmetic. It was then conjectured by A Selberg that there were no continuous deformations of these 'higher rank' discrete groups. This was proved by Selberg in many special cases and by Andre Weil in general using methods of Calabi–Vesentini. This lack of deformation may be viewed as saying that Γ is locally rigid: all nearby deformations are conjugate to each other. The Weil rigidity theorem was extended to non-cocompact discrete groups (of rank one) by Garland and Raghunathan.

George Mostow discovered a much stronger property for these Γ; he termed this 'strong rigidity': it says that if two cocompact discrete groups in two different simple Lie groups G and G' are isomorphic as abstract groups, then any abstract isomorphism arises from an isomorphim of the ambient Lie groups G and G', provided G has dimension >3. This strong rigidity was extended to rank one non-cocompact discrete groups by Gopal Prasad [15]. The proofs of these results started with the observation that a map of lattices gave rise to an equivariant quasi-isometry between the symmetric spaces, which gave a map between the boundaries of these symmetric spaces. The boundary turns out to be the Tits building and therefore the theorem of Tits on maps between buildings being induced from homomorphisms of the groups yields the strong rigidity theorem.

It had been noted (by Selberg, for instance) that the only general method of constructing cocompact (more generally, finite covolume) discrete subgroups in higher rank groups was 'by arithmetic'.

In order to explain that, we need to define the notion of an 'arithmetic subgroup' of an algebraic group. We will in fact define a somewhat more general concept (as it will be needed later). Let k be a field and K an algebraic closure of k. Let G be an algebraic subgroup of $GL_n(K)$ defined over k (or simply a k-algebraic subgroup). This means that G is a subgroup of $GL_n(K)$, which is also the set of zeroes of a collection of polynomials in the entries of elements of $GL_n(K)$ and the inverse of the determinant function on $GL_n(K)$, all of which have coefficients in k. The k-points $G(k)$ of G is the intersection of G with $GL_n(k)$.

Suppose now that k is a global field (number fields are global fields) and S is a (finite) set of valuations of k containing all the Archimedean valuations of k. For a non-Archimedean valuation v of k, we denote by k_v the completion of k at v, and by O_v the ring of integers in k_v. Also for S as above, O_S is the ring of S-integers in K: O_S is the set of all x in k which are in O_v for all non-Archimedean v outside S. We denote by $G(O_S)$ the subgroup of all elements x in $G(k)$ such that x and x_{-1} are both in $GL_n(O_S)$. A subgroup Γ of $G(k)$ is defined to be an S-arithmetic subgroup of G if $\Gamma \cap G(O_S)$ has finite index in Γ as well as in $G(O_S)$. When S is precisely the set of Archimedean places, an S-arithmetic subgroup will be called an arithmetic group.

For each v in S, the group $G(k_v)$ has a natural locally compact topology deduced from that on k_v. We denote by $\mathbf{G}_S$ the direct product of the $G(k_v)$ as v runs through S. One then has a natural diagonal inclusion of $G(k)$ in $\mathbf{G}_S$, and hence also of any S-arithmetic subgroup $\Gamma \subset G(k)$. With this notation we have the following fundamental result due to Borel and Harish-Chandra.

Theorem 3. *Suppose that G is semisimple (this means that the Lie algebra of G is semisimple). Then Γ has finite covolume in $\mathbf{G}_S$.*

When S is precisely the set of Archimedean valuations, $\mathbf{G}_S$ is a real Lie group (k_v is the field of real or complex numbers for all V in S), so that we obtain in this fashion discrete subgroups of semisimple Lie groups.

For certain choices of G and k, it turns out that the projection of Γ on one factor $G(k_v)$ of $\mathbf{G}_S$ is a discrete cocompact subgroup of that factor. Borel used this to show that one can construct cocompact discrete subgroups in any semisimple real Lie group.

One says that a discrete subgroup Γ of a semisimple Lie group H is arithmetically defined if the following holds: let H' be the quotient of H by its centre. Then there is a number field k, a semisimple algebraic group G over k and an arithmetic subgroup Γ' of G, such that H' is isomorphic (as a real Lie group) with a factor of the product of $G(k_v)$, where v runs through all Archimedean places of k;

further, the image in H' of the discrete subgroup Γ of H, and the projection of the arithmetic group $\Gamma' \subset \prod_v G(k_v)$ on H', are commensurable in H', i.e. their intersection has finite index in both. This seems a rather technical definition, but it is necessary when working in the general context in which the results naturally find their place.

Selberg and Pjatetskii–Shapiro conjectured that all discrete subgroups with finite covolume in a semi-simple Lie group are arithmetically defined (in the technical sense explained abvove). It was proved for a fairly substantial class of discrete groups (to be specific, for the class of discrete groups Γ in groups with $\mathbb{Q}$-rank ≥ 2) by M S Raghunathan, and proved by G A Margulis in full generality.

Margulis discovered an extremely strong generalisation of rigidity (which was termed 'superrigidity' by Mostow). It says that any linear representation (over any locally compact field) of the discrete group Γ extends essentially to the whole Lie group G, provided G has $\mathbb{R}$-rank at least two – the $\mathbb{R}$-rank of G is characterized as the dimension of a maximal connected abelian subgroup A of G, such that no non-trivial element of A is a unipotent element of G (unipotent elements are those which act on the Lie algebra with all eigenvalues equal to 1).

This phenomenon of super rigidity was known – at least for *arithmetic* Γ obtained from $\mathbb{Q}$-groups with $\mathbb{Q}$-rank ≥ 2 – using the congruence subgroup property (see section 5.4 below), by the work of Bass–Milnor–Serre and M S Raghunathan: in this case, the authors exploited the existence of unipotent elements in Γ.

Margulis showed, however, that super rigidity holds for any higher rank lattice Γ; the proof used ideas from diverse fields and a crucial component was the use of ergodic theory and measure theory in establishing essentially algebraic properties of lattices. Remarkably, he showed that the argument can be turned around: that superrigidity implies the Selberg–Pjatetskii Shapiro conjecture on arithmeticity.

These results were extended to the positive characteristic case by T N Venkataramana [27].

5.2 *Cohomology*

The theorem of Weil referred to above may be stated differently. Denote by $H^i(\Gamma, \rho)$ the i-th cohomology group of the discrete group Γ with coefficients in a finite dimensional representation ρ. The local rigidity theorem of Weil follows from the vanishing result (for the 'adjoint representation' Ad):

$$H^1(\Gamma, \mathrm{Ad}) = 0.$$

This vanishing result was greatly generalised by Matsushima for the trivial representation and Raghunathan in general: the result states that if Γ is a lattice in a simple Lie group G other than $SO(n,1)$ or $SU(n,1)$, and ρ is any finite dimensional representation of G, then

$$H^1(\Gamma, \rho) = 0.$$

Even in the case of $SO(n,1)$ and $SU(n,1)$ the vanishing theorem holds, except for a special class of ρ.

There is a general formula, called the Matsushima formula, which relates cohomology of cocompact discrete groups to the occurrence in $L^2(\Gamma\backslash G)$ of certain 'cohomological representations' of G. The construction and classification of these representations was almost completely solved by Parthasarathy and Kumaresan. A much more general result was proved by Vogan and Zuckerman.

When the arithmetic discrete group is no longer compact, the analogue of the Matsushima formula becomes much more difficult to establish. Even to prove vanishing of cohomology, one must relate the cohomology to suitable square summable forms. This was accomplished in [19]. A recent theorem of Jens Franke shows that cohomology is represented by automorphic forms. A description of the 'weighted L^2-cohomology' of a non-cocompact arithmetic group was achieved by Arvind Nair [10].

5.3 *Normal subgroups*

Bass, Milnor and Serre discovered that if $n \geq 3$, then any infinite normal subgroup of $SL_n(\mathbb{Z})$ was of finite index. This theorem was extended by Raghunathan to any non-cocompact arithmetic irreducible subgroup of a semi-simple group of $\mathbb{R}$-rank greater than one. Using ergodic theoretic techniques, Margulis proved that if Γ is an irreducible lattice Γ in any higher rank semi-simple Lie group G, then all infinite normal subgroups are of finite index (extending the result of Raghunathan to the cocompact case). In the course of proving this, Margulis proved that any Γ-equivariant measurable quotient of the variety G/P (P a minimal real parabolic subgroup) is actually G/Q with $Q \supset P$ a parabolic subgroup. Dani proved a continuous version of this result, answering a question of Margulis. Thus, classifying infinite normal subgroups is equivalent to classifying normal subgroups of finite index in 'higher rank' arithmetic groups.

5.4 *The congruence subgroup problem*

We now turn to classification of finite quotients of arithmetic groups. Let us start with an example: the group $\Gamma = SL_2(\mathbb{Z})$ of 2×2-matrices with integer coefficients and determinant one. This has many subgroups of finite index; one way of obtaining finite quotients of $SL_2(\mathbb{Z})$ is to consider an integer $k \geq 0$ and the attendant ring homomorphism $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$; this induces a homomorphism of groups $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/k\mathbb{Z})$. The image is surjective and is a finite quotient. The kernel consists of matrices, which are congruent to the identity matrix modulo k, and is a subgroup $\Gamma(k)$ of finite index. This is called a principal congruence subgroup of level k. A subgroup of $SL_2(\mathbb{Z})$ which contains a principal congruence subgroup $\Gamma(k)$ is called a congruence subgroup of $SL_2(\mathbb{Z})$.

The question is whether every finite index subgroup of $SL_2(\mathbb{Z})$ is a congruence group; this turns out to be false for $SL_2(\mathbb{Z})$ but holds true for groups like $SL_3(\mathbb{Z})$. One can ask a more general question. Let k be a global field and S a finite set of places including all Archimedean places. Let O_S be the ring of S-integers in k. Let G be a linear algebraic subgroup of GL_n defined over k. The group $G(k)$ of k-points of G can be made into a topological group by declaring the set of all S-arithmetic subgroups (e.g., section 5.1) as a fundamental system of neighbourhoods of the identity. The completion of $G(k)$ (with respect to the left uniform structure) is a locally compact group, which we denote by $\mathbf{G}_a$ in the sequel. $G(k)$ can also be topologised by declaring the set of all S-congruence groups to be a fundamental system of neighbourhoods of the identity: an S-arithmetic subgroup is an S-congruence subgroup if it contains the kernel of the natural homomorphism of $G(O_S)$ in $GL_n(O_S/I)$, where I is a non-zero ideal in O_S. The corresponding completion of $G(k)$ is denoted $\mathbf{G}_c$. One then has a natural map of $\mathbf{G}_a$ in $\mathbf{G}_c$ whose kernel we denote by $C(S,G)$. Then $C(S,G)$ measures the failure of S-arithmetic subgroups to be S-congruence subgroups.

Much is known about $C(S,G)$ for G isotropic over k. The following result is due to Raghunathan [20,21], and subsumes earlier results due to Bass–Serre–Lazard, Mennicke, Bass–Milnor–Serre, Matsumoto, Vasserstein and Bak–Rehman; [20] deals with the case of groups of k-rank 1, and makes use of earlier results due to Serre on SL_2.

Theorem 4. *Assume that G is simply connected, isotropic over k and that the sum of the k_v-ranks of G as v varies over S is at least 2 (where k_v is the completion of k at v). Then the group $\mathbf{G}_a$ is a central extension of $\mathbf{G}_c$, and $C(S,G)$ is isomorphic to a subgroup of the group of roots of unity in k (in particular, it is a finite cyclic group).*

The same assertion holds also for a number of anistropic G as well. For spin groups, this is due to Kneser. His techniques were generalised by Rapincuk and Tomanov (independently) to cover groups of type D_n and some others.

The centrality of $C(S,G)$ is the main thrust of [20] and [21]. The structure of $C(S,G)$ is then a consequence of some cohomology computations due to Prasad and Raghunathan [16,17]. These last results are generalisations of theorems about central extensions of p-adic split and quasi-split groups due respectively to C C Moore and V Deodhar.

B Sury solved the congruence subgroup problem for a large class of S-arithmetic groups, where S is the complement of a finite set; this generalised a theorem of Margulis.

6. Algebraic groups, groups over other fields

6.1 *Classification*

Most Lie groups are the groups of $\mathbb{R}$-points of $\mathbb{R}$-algebraic groups (e.g., section 5.1); some examples are $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_n(\mathbb{C})$. A classification of k-algebraic groups is possible over fields k like $\mathbb{R}$, $\mathbb{C}$ and local fields. Over number fields, the classification depends on the 'Hasse principle' which says essentially that if two groups are isomorphic over all completions, then they are isomorphic. The classification over other fields is not completely known. Over certain special fields, there is a conjecture of Serre which describes the classification in terms of vanishing of Galois cohomology. Serre's conjecture was proved for all classical groups by R Parimala and Fluckiger.

A related question is the classification of principal G-bundles and was settled by Raghunathan, Ramanathan and in many special cases, by Parimala.

Recently Suresh has proved the related conjecture that all quadratic forms over certain fields represent zero.

6.2 *Kac–Moody groups and algebras*

We have already mentioned work on loop groups. These loop groups are closely connected to groups over rings of the form $\mathbb{C}[t, t^{-1}]$ (the Laurent series ring in one variable. These are called Kac–Moody Lie groups (with associated Lie algebras); their representation theory has remarkable similarities with the finite dimensional case, and yet there are subtle differences. Much pioneering work was done by V Chari and A Pressley, and S E Rao has continued work on these and related topics.

7. Connections with ergodic theory

One of the most active areas of research is the interface between Lie groups and ergodic theory. The fundamental theorem which relates these two fields is the Howe–Moore ergodicity theorem.

Theorem 5. *If $\rho : G \to U(H)$ is a representation of a non-compact simple Lie Group G on a Hilbert space H, and $v \in H$ is a non-zero vector which is fixed under a non-compact subgroup U of G, then all of G fixes the vector v.*

An immediate consequence is that if $H = L^2(\Gamma\backslash G)$, then every non-compact closed subgroup U of G acts ergodically on $\Gamma\backslash G$. This theorem is already used crucially in the arithmeticity theorem of Margulis alluded to above. In particular, a unipotent subgroup U acting on $\Gamma\backslash G$ acts ergodically. As a consequence, almost all orbits of U are dense.

A major driving force in the area was the conjecture due to Dani and Raghunathan, that orbit closures of connected unipotent groups acting on $\Gamma\backslash G$ can be completely described (the last sentence of the previous paragraph says that almost all orbits are dense). A substantial progress on the conjecture was made by Dani [2] and later by Shah [24]. A major motivation for the conjecture was a conjecture in number theory due to Oppenheim, which says that the values of 'irrational' quadratic forms on the integral lattice come arbitrarily close to zero. This can be reformulated (the reformulation is due to Raghunathan) in terms of unipotent orbits as described above. The Oppenheim conjecture was proved by Margulis.

The Dani–Raghunathan conjecture was proved in full generality by Marina Ratner [23].

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