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Some highlights on the work in probability theory in India during 1980–2008: A report

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We are considering a period of approximately 25 years for this report. Since most of the work during this period has been in the area of applied probability, stochastic processes and quantum probability, we restrict ourselves to these themes.

1. Applied probability

A considerable amount of research in applied probability has been going on in the universities and the IITs where the focus has been on topics like queuing theory, reliability theory, branching processes, and applications to operations research. Some of the strands in this have been: explicit calculations for transient response of queues, optimization problems in manufacturing and inventory control, numerical methods such as matrix analytic schemes and so on. Some major centres of such activity have been the groups around Professors J Medhi in Guwahati, U N Bhat in Dharwad, A Krishnamoorthy in Cochin, O P Sharma in IIT, Delhi, M N Gopalan in IIT, Mumbai, P R Parthasarathy and N Rangan in Chennai. This direction of research has led to the discovery of some interesting connections between pure mathematical ideas like continued fractions, orthogonal polynomials and tridiagonal matrices on the one hand and Markov processes arising naturally from the study of birth and death processes, growth of cancer cells, queuing theory and congestion in networks on the other. Some of the interesting books that have appeared in this context are:

- J Medhi, *Stochastic processes*, Wiley Eastern, New Delhi 1982;
- U N Bhat and G K Miller, *Elements of applied stochastic processes*, John Wiley, New York 2002;
- R B Lenin and P R Parthasarathy, *Birth and death models with applications*, American Science Press Inc., Syracuse, N Y 2004.

Since the 1980s there has been much activity in controlled Markov chains and diffusion processes. Topics like ergodic and risk-sensitive control, control under constraints, partially observed control problems, control of switched diffusions, adaptive control, stochastic dynamical games, optimization problems in insurance modelling viewed as a multi-person dynamic game, the role of the Skorohod problem in the search for Nash equilibrium, treatment of Hamilton–Jacobi–Bellman equations with discontinuous Hamiltonians, probabilistic approach to solutions of 1st order nonlinear differential equations, etc. have received much attention and several interesting results have appeared from the papers of V S Borkar of TIFR, M K Ghosh of IISc, K Sureshkumar of IIT, Mumbai, Abhay Bhatt of ISI, Delhi, S Ramasubramanian of ISI, Bangalore and so on. Applications to the allied fields of nonlinear filtering, mathematical finance, stochastic approximation theory and problems of simulations and computations have been considered by Professors R L Karandikar (Cranes Software Co), A Bhatt of ISI, Delhi, G Basak of ISI, Kolkata and S K Juneja of TIFR. Some of the

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important books that have come out of this effort are:

- G Kallianpur and R L Karandikar, *White noise theory of prediction, filtering and smoothing*, Gordon and Breach, New York 1988;
- ibid: *Introduction to option pricing theory*, Birkhauser-Verlag, Boston 2000;
- V S Borkar, *Topics in controlled Markov chains*, John Wiley, New York 1991;
- ibid: *Stochastic approximation: A dynamical systems viewpoint*, Hindustan Book Agency, New Delhi 2008.

2. Stochastic processes and limit theorems

The most interesting developments have centred around asymptotics of diffusion processes, reflected diffusions, connections with partial differential equations, Skorohod problem, criteria for the existence of invariant measures for Markov processes characterized by martingale problems, continuum percolation models, uniqueness for martingale problems arising from super-processes, infinite dimensional stochastic differential equations and branching-coalescing particle systems and their ergodic properties.

S Ramasubramanian (ISI, Bangalore) and collaborators have obtained usable criteria for testing recurrence, transience, and positive recurrence for diffusions and reflected diffusions in terms of Lyapunov-type functions. By similar methods, (d-2)-dimensional submanifolds have been shown to be polar sets for d-dimensional diffusions and reflected diffusions. A functional central limit theorem has been established for diffusions with almost periodic coefficients. A probabilistic treatment of Neumann and oblique derivative boundary value problems in terms of reflected Brownian motion and reflected diffusions has been developed. Using a combination of analytic and probabilistic methods, transition probability functions of different kinds of reflected diffusions have been explicitly constructed. Collaboration with Amites Dasgupta has resulted in a similar construction of transition probability densities for reflected symmetric stable processes.

Srikanth Iyer (IISc, Bangalore) has studied several aspects of measure-valued diffusions. In particular super-processes with self-attraction have been constructed by an absolutely continuous change of measure technique using the occupation measure and self-intersection local times. Siva Athreya (ISI, Bangalore) and Abhay Bhatt (ISI, Delhi) have studied infinite dimensional diffusions extensively by appealing to martingale problems and the famous Stroock–Varadhan theory.

The continuum percolation set-up consists of d-dimensional balls of random radius centered at points of a Poisson process with a fixed intensity in a d-dimensional space. The model is parameterized by the intensity and the random radius mentioned above. For a given value of this random radius consider the smallest intensity (i.e., the critical intensity) needed to obtain a percolating cluster, i.e., for the existence of an unbounded connected component in the region covered by the balls. For a long time it was believed by the physicists that the average volume covered by the balls in a unit cube parameterized the model. In his PhD thesis, Anish Sarkar has a result which shows why this belief is wrong.

In addition, questions connected with the geometry of the covered regions have been examined. Studies were conducted on problems regarding the number of unbounded clusters in the super-critical region as well as the relation between the critical intensities for different shapes other than balls but of the same volume. The book: Ronald Meester and Rahul Roy, *Continuum percolation*, Cambridge University Press, 1996 gives an account of the investigations cited above and several unpublished results.

Beginning 2000, a group of probabilists including Rahul Roy, Anish Sarkar, Siva Athreya, Abhay Bhatt, Amites Dasgupta, Sreela Gangopadhyay and others have been studying random graphs motivated by the desire to understand the physical theory of drainage networks. These networks consist of nodes randomly placed in a d-dimensional Euclidean space. Each node is connected to its nearest node in a given direction. Local asymptotic properties of this model have been studied when there are n nodes placed in a unit square. They are connected with the Dickman function of number theory. Globally it has been shown that in 2 and 3 dimensions the random graph is a tree whereas it is a forest in higher dimensions. Under an appropriate scaling drainage networks have been shown to obey a functional central limit theorem in the form of convergence to a Brownian web.

Kavi Ramamurthy (ISI, Bangalore) and collaborators have obtained an explicit solution to the prediction of continuous time p-stable processes by reducing it to an extremum problem in the Hardy space of p-th order in the upper half plane.

Arup Bose and colleagues have developed a unified approach for obtaining the limiting spectral distribution of general real symmetric patterned random matrices along with a rate of convergence of the empirical spectral distribution to this limit. A large number of limit theorems for various probabilistic models have been established by Amites Dasgupta, Arup Bose, Arindam Sengupta, Sreela Gangopadhyay, Krishanu

Maulik, Gopal Basak, B V Rao, Atanu Biswas, Santanu Chakraborty and others with a wide variety of applications. Nabin Jana and B V Rao have studied the problem of existence of free energy in random energy models of statistical mechanics and phase transitions. They have applied successfully Varadhan's theory of the large deviation principle in arriving at their beautiful solution. In this context of limit theorems, it is of interest to note the contribution of Siva Athreya to the existence of the infinite volume limit for the stationary distribution of abelian sandpile models.

In the context of statistical methods as applied to stochastic processes an immense amount of work has been done under the leadership of B L S Prakasa Rao. We expect this work to be covered in the report on the progress of statistics in Indian institutions during the period relevant for the platinum jubilee of the academy. However, it is important to highlight the following books published by B L S Prakasa Rao of ISI, Delhi:

- *Semimartingales and their statistical inference*, Chapman and Hall, London 1999;
- *Statistical inference for diffusion type processes*, Arnold, London 1999;
- *Identifiability in stochastic models; characterization of probability distributions*, Academic Press, Boston 1992.

3. Quantum probability

In 1984, R L Hudson and K R Parthasarathy established a quantum version of Ito's classical formula for the standard Brownian motion and showed how quantum dynamical semigroups describing irreversible processes are obtained by coarse-graining unitary Schrodinger-type evolutions of open systems, i.e., quantum systems placed in a bath. Through their seminars to a group of PhD students, Kalyan B Sinha and K R Parthasarathy wrote a series of papers laying the foundations of a quantum stochastic calculus treating topics like quantum stop times, martingale representation theorems and models of quantum Markov processes. K B Sinha and Anilesh Mohari showed how quantum stochastic differential equations with infinite degrees of freedom could be solved. This enabled them to describe the Evans–Hudson flows with infinite degrees of freedom and also indicate how equations with unbounded operator coefficients could be solved in some cases. B V Rajarama Bhat and K R Parthasarathy showed how quantum Markov processes could be constructed starting from a quantum dynamical semigroup and the fruitful notion of a strong quantum Markov process introduced. B V R Bhat made a fruitful connection between this theory and the theory of endomorphism semigroups on the algebra of all operators on a Hilbert space. Kalyan B Sinha and Debashis Goswami developed the quantum stochastic calculus by exploiting the theory of Hilbert modules. Kalyan Sinha and Lingaraj Sahoo strengthened the convergence of quantum random walks to quantum diffusions. Finally, an interesting link between quantum stochastic calculus and noncommutative geometry was established. Two important books that have come out of the work by this group are:

- K R Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhauser Verlag, Basel 1992;
- D Goswami and Kalyan B Sinha, *Quantum stochastic processes and noncommutative geometry*, Cambridge University Press, 2007.