# Partial differential equations

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This section is divided into seven subsections which present various aspects of partial differential equation (PDE) highlighting the contributions by Indians with appropriate references. I sincerely thank my colleagues who have enthusiastically prepared the subsections: G D V Gowda, K T Joseph, S Kesavan, P Prasad, M Ramaswamy, P N Srikanth and A S Vasudeva Murthy.

## I Elliptic PDEs

One of the most significant Indian contributions to the theory of elliptic partial differential equations is the work of S Minakshisundaram. His work resulted in the genesis of a fertile research area, known today as geometric spectral asymptotics. Geometric spectral asymptotics is the study of the relationship between the coarse features of the geometry of a domain and those of the spectrum of the Laplace operator. There was a flood of developments in this area in the 1970s. Together with A Pleijel [13], he introduced a function known as the Minakshisundaram–Pleijel zeta function, analogous to the famous Riemann zeta function. The residues at the poles of this function give information about the averaged density of the eigenvalues in the high frequency limit and also on eigenfunctions.

To just cite one application of this work, we recall the famous question of Marc Kac, 'Can one hear the shape of a drum?' In mathematical terms, this translates to the question whether two domains which have the same spectrum for the Laplace operator (they are then called isospectral domains) are congruent to each other. This problem, posed around 1966 [3], was settled, negatively, in the 1990s. However, using the work of Minakshisundaram and Pleijel, done in the 1950s, it can be shown that the spectrum fixes the perimeter of a two-dimensional domain. That it also

fixes the volume, comes from a celebrated work of Weyl done in the early part of the twentieth century. Thus, using the classical isoperimetric inequality, we immediately see that, if one of two isospectral domains is a disc, so is the other. Thus, we can definitely 'hear' the shape of a circular drum!

Another milestone in the theory of elliptic equations is a theorem by T Kotake and M S Narasimhan [12]. This theorem gives a characterization of analytic functions in terms of bounds for the action of powers of an elliptic operator with analytic coefficients on it.

The 1960s saw the growth of a new area in the calculus of variations known as elliptic variational inequalities. This relates to the optimization of energy functionals related to elliptic operators. One of the pioneers in this area was G Stampacchia. A lot of important work on elliptic variational inequalities was done by M K Venkatesha Murthy and G Stampacchia [15].

Murthy and Stampacchia also authored a comprehensive paper [14] on degenerate elliptic equations with special emphasis on existence, global regularity, interior local regularity, local boundary regularity and Hölder continuity of solutions.

Up to the 1970s, we thus see that the Indian contributions to the theory of partial differential equations in general, and that of elliptic equations in particular, while having been quite significant, have been rather sporadic. With the establishment

of the TIFR-IISc programme in the Applications of Mathematics at Bangalore in the 1970s, there has been a regular stream of high quality work done in these areas. Talking of elliptic partial differential equations, important contributions have been made in the areas of elasticity, control of systems governed by elliptic equations, homogenization, isoperimetric inequalities and semilinear boundary value problems.

Mathematical elasticity studies the deformation of three dimensional elastic bodies. A special class of such bodies are thin elastic bodies such as plates, shells, rods, etc. Engineers and physicists have used, with great success, approximate lower dimensional (two, in the cases of plates and shells and one, in case of rods, for example) mathematical models to study such bodies. These models have been obtained, in a substantial measure, by heuristic and physical arguments. In the late 1970s, work started in France to study such models in a systematic manner. For example, a plate is considered as a three dimensional cylindrical body, with a very small thickness (i.e. height) $\varepsilon$. The relevant three-dimensional equations are written down in this domain, and then by a simple scaling the domain is transformed to be the corresponding cylinder of unit height. Rewriting the equations in the new (fixed) domain brings out the thickness $\varepsilon$ as a parameter and a formal asymptotic expansion of the solution was written with respect to this parameter. An analysis of the leading term shows it to be the solution of a well-known two-dimensional plate model used by engineers. All this is formal, but was still quite satisfactory for linear traction problems. However, when studying vibrations of elastic plates (eigenvalue problems) the need for a convergence theory arose due to the existence of an infinite sequence of eigensolutions. The first such convergence result, which justified the two dimensional model with full mathematical rigour was due to S Kesavan and P G Ciarlet [1]. This was then systematically used to justify mathematically a large number and variety of lower dimensional models in elasticity theory. In India, special mention must be made of the work done in connection with vibrations of elastic structures (shells [9,10] and rods [11]).

The subject of isoperimetric inequalities is a very interesting one and is the meeting point of analysis, geometry and partial differential equations. One of the oldest problems in this area goes back to ancient Greece and is known as Dido's problem: how should one place a given closed rope on the ground so as to enclose the maximum possible area? In mathematical terms, 'amongst all simple closed curves in the plane of given perimeter, find that which maximizes the enclosed area.' The answer is that it is the circle, and only the circle. If $L$ is the perimeter of a curve and $A$ is the enclosed area, then we have the classical isoperimetric inequality (alluded to earlier when referring to the work of Minakshisundaram)

$$L^2 \geq 4\pi A.$$

Equality occurs only for the circle. Today, by an isoperimetric problem, we mean an optimization problem for a shape-dependent functional, keeping some geometric parameter fixed. The first such problem involving the 'energy' of the solution of an elliptic partial differential equation was that of the torsion of an elastic beam posed by Saint Venant in the 1850s and solved by Polya in the twentieth century. Lord Rayleigh, in his book '*Theory of sound*', conjectured (around 1894) that of all drums of fixed area, the circular drum has the lowest fundamental frequency. In the language of partial differential equations, it states that of all (two-dimensional) domains of fixed area, the circle is such that the Laplace operator has the least first eigenvalue (under homogeneous Dirichlet boundary conditions). This was settled independently in the 1920s by Faber and Krahn. The result is also true in all space dimensions.

One of the techniques for studying such problems (especially when the sphere is the expected optimal shape) is that of Schwarz symmetrization (cf. the book by S Kesavan [4], for a recent account of this subject). Symmetrization also provides comparison theorems for solutions of partial differential equations with those of some 'model' equation over a ball which will usually have a radially symmetric solution that can be computed by hand. A pioneering result in this direction was due to Talenti. The equality case for comparison theorems of elliptic equations (linear and nonlinear) and systems was studied by S Kesavan and collaborators [2,5–8]. One of the off shoots of these studies was also the establishing the radial symmetry of solutions of certain nonlinear equations and systems in a ball, using techniques from symmetrization and nonlinear functional analysis. Usually such results are proved for domains possessing symmetry about an axis using maximum principles.

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## II Nonlinear elliptic PDEs

### 1. *Introduction*

The analysis of physical models which was the prime motive to study differential equations started in 18th century and has remained to the present day one of the concerns of the development of PDE's in general and as well in the more specific case of elliptic problems. The other major motive in this specific case comes from differential geometry, especially in its global aspects. Outstanding developments in the linear theory which came about in the second half of 20th century, *a priori* estimates, interpolation properties of Sobolev spaces, maximum principles, etc. made the calculus in Sobolev spaces an important tool to carry forward the studies into the realm of nonlinear problems. In the contemporary scene, major studies are concerned with nonlinear problems. In the next section we will highlight those problems where Indian mathematicians have made significant contributions.

### 2. *Nonlinear problems*

A typical model in which all the questions handled can be encompassed is:

$$(P_\lambda): \quad Lu = f(x, \lambda, u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega.$$

Here $L$ is in most cases $-\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ or $\Delta^2$ or the $p$-Laplacian: $\nabla \cdot (|\nabla u|^{p-2}\nabla u)$, $\Omega \subseteq \mathbb{R}^n$ is a bounded or unbounded domain and $\partial\Omega$ is the boundary, with $B$ a boundary operator. When $\Omega$ is unbounded, the boundary conditions are to be understood appropriately. The function $f$ is a real valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}$. Typical questions concerning $(P_\lambda)$ are:

1. Existence/nonexistence of solutions of $(P_\lambda)$ depending on the hypothesis on $f$. In most cases, motivated by how such model problems came about, the question of existence is restricted to positive solution.
2. Multiplicity of solutions; here the question is again linked to the context (i.e., nature of $f$) where the solutions need to be of particular sign or could infact change sign.
3. Issues concerning symmetry/break of symmetry of solutions. In such a context $\Omega$ is typically a ball or an annulus.
4. Issues concerning understanding Palais–Smale sequences, concentration phenomena, typical to non-compact problems.
5. Questions concerning uniqueness of solutions.
6. Behaviour of solutions as $\lambda \to \infty$ when certain scaling is possible.

It will be impossible to keep the article to any reasonable length if one wants to motivate the hypothesis one makes on $f$ highlighting the importance of the questions posed, however to put the results in proper perspective we will give some details as we go along.

Typical approaches to studying the problem use tools from topological & variational methods such as degree theory, Morse theory (index theories), bifurcation theory (global), mountain pass theorem (MPT) of Ambrosetti–Rabinowitz,

symmetry results of Gidas–Ni–Nirenberg, perturbation theory, etc. We will now divide the section into subsections and explain a few results.

## 3. *Non-compact problems*

To motivate the results, we begin with a particular example and build around it all other examples. Let $\Omega \subset \mathbb{R}^3$ and $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. Consider the problem

$$(P): \quad -\Delta u = u^p \ \text{ in } \ \Omega,$$
$$u > 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ in } \ \partial\Omega.$$

If $1 < p < 5$ MPT guarantees the existence of a solution. The MPT is used in analysing the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{(p+1)} \int_\Omega (u^+)^{p+1}, u \in H_0^1(\Omega).$$

All the geometric hypothesis required in MPT are valid even if $p = 5$ but a crucial compactness hypothesis namely the Palais–Smale condition fails to hold and this is due to the fact that the inclusion of $H_0^1(\Omega)$ (with $\Omega$ bounded) in $L^{2^*}(\Omega)$ is not compact, where $2^* = 2n/(n-2)$, whereas it is compact in $L^{2^*-\varepsilon}(\Omega)$ for every $\varepsilon > 0$. In fact $(P)$ has no solution when $p = 5$, a fact which follows from Pohozaev's identity. Beginning with the epoch making work of Brezis and Nirenberg [1] problem $(P_\lambda)$, where $f$ has appropriate growth, making it a non-compact problem in dimension $n \geq 3$ received a great deal of attention with $Bu = u$ on $\partial\Omega$. These developments naturally lead to more and more questions, namely how to characterize non-compact problems in $n = 2$ and in $n \geq 3$ with more general $B$, more specifically the Neumann boundary data.

In [2] the authors came up with an example of an $f$ in which the nonexistence of a positive solution was proved when $\Omega \subset \mathbb{R}^2$ and the choice of $f$ was dictated by Moser–Trudinger embedding theorem. This lead to further work on characterizing the border line problem in $\mathbb{R}^2$ and a complete description was obtained in a series of papers by Adimurthi and his collaborators in [3–8].

The questions concerning Neumann boundary problems in the non-compact case have been comprehensively studied in series of excellent papers bringing out role of mean curvature or more generally geometry of the boundary [9–15]. The other class of non-compact problems where a very substantial and very significant contribution has come

from Indian mathematicians is concerned with the Hardy–Sobolev operator. Some of the important contributions here are [16–21]. A very significant contribution on the Brezis–Nirenberg type problem is contained in [22].

## 4. *Multiplicity and exactness*

During the early eighties problems of the type

$$(P_t): \quad -\Delta u = f(u) \pm th \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega$$

with typical hypothesis of the type $\lim_{s\to\pm\infty} \frac{f(s)}{s} = \lambda_\pm$ were studied in great detail for existence of multiple solutions as the parameter $t$ varies and $h$ is a given function. This is the so called Ambrosetti–Prodi type problem. The general conjecture was $(P_t)$ admits at least $2k + 1$ number of solutions, where $k$ is the number of eigenvalues in the interval $(\lambda_-, \lambda_+)$. Given the nature of the nonlinearity it was conjectured that if $f(s) = s^2$ then for any $N > 0$, there exists $t_N$ such that $(P_t)$ has at least $N$ solutions for $t > t_N$. This conjecture remained hopelessly open till the work by Srikanth and collaborators [23]. The result in [23] exploited the topological information of mountain pass solutions through Morse index and in a way also provided a new way of looking at break of symmetry of solutions. Other significant contributions in the area which lead to further studies in related questions are [24,25].

## 5. *Break of symmetry*

In a celebrated work, Gidas–Ni–Nirenberg proved that if $u$ solves

$$-\Delta u = f(u) \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and $\Omega$ is a bounded ball in $\mathbb{R}^n$, then $u$ is radial. Beginning with this result and motivated by other considerations, questions concerning break of symmetry of solutions have been of high priority. Very significant contributions from Indians are contained in [26,27].

## 6. *Uniqueness*

In general, the question of proving existence of uniqueness for a problem like

$$(P_\lambda): \quad -\Delta u = f(u, \lambda) \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

is very difficult and a very important contribution in this direction is in the paper by Srikanth [28].

Here certain non-degeneracy has been established for solutions with Morse index one. This has a potential for applications to other problems and is being carried out by many mathematicians working both in India and elsewhere. Other significant contributions by Indian mathematicians are in [29] and [30].

## 7. *Others*

There have been some very interesting contributions in studies concerning $p$-laplacians the biharmonic operator, and results concerning symmetry of solutions in the spirit of the work of Gidas–Ni–Nirenberg, etc. No effort is being made in this writeup to explain their significance in which case the article would go beyond resonable length.

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## III Homogenization in PDEs and related developments

Multiscale problems in science and engineering are very common. In this class of problems, the unknowns i.e., the quantities of interest exhibit variation on a wide range of scales and this poses enormous challenge to their computation. Many fluid flow problems fall in this class. To overcome this difficulty, many methods have been devised. (The range of applicability of each one of them is rather limited; general problem remains open). One of them is homogenization. The idea is to focus on macro features of the solution and one tries to obtain an approximation for its large scale component by averaging out small scales. How to carry out this task? Of course, the difficulty is that all scales may be coupled and they may interact among each other creating the so-called 'microstructures.' Thus there is a need to describe how a given system can propagate oscillations and transfer energy from one scale to another through interaction. One may think of two classes of systems: one which creates oscillations even though they are not present in the data and propagates them; another one which does not create but simply propagates oscillations from the data. (By data, we mean initial, boundary data, domain, coefficients, source term, etc.) Obviously, treatment of the first class of systems is harder than the second one. In homogenization theory, one strives to develop tools and methods to investigate this aspect of systems. This is in sharp contrast to the classical approach which seeks the effect of regularity of data on the solution (rather than the effect of the singularity/oscillations in the data). Though mathematical justification of turbulence modeling remains a dream in the subject of homogenization, less complicated problems originating from applications drive various developments of the subject.

In this brief write-up, we begin with an example in which we see explicitly the effect of oscillations of the coefficients on the solution. In the subsequent discussion, instead of presenting a variety of results of homogenization, we mention several approaches, generalizations and tools with appropriate references in the literature. Their strength and weakness are mentioned along with examples of the homogenization problems they can solve.

### 1. *Method of two scale asymptotic expansion*

Let us consider the problem of steady state heat conduction in rough media. This is modeled by a linear, second order, scalar elliptic equation with periodic coefficients with periodic cell $Y$. By decreasing the period, we give more oscillations to the coefficients and increase the complexity of the medium. These periodic coefficients are said to define microstructure in this problem. More precisely, let us consider the operator

$$A \equiv -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial}{\partial y_l}\right), \quad y \in \mathbb{R}^d,$$

where

$$\begin{cases} a_{kl} = a_{lk}, |a_{kl}(y)| \leq M, \\ a_{kl}(y) \text{ is } Y-\text{ periodic with } Y = \prod_{j=1}^{d} ]0,1[, \\ \text{there exists } \alpha > 0 \; a_{kl}(y)\eta_k\eta_l \geq \alpha|\eta|^2, \\ \forall y \in \mathbb{R}^d, \; \forall \eta \in \mathbb{R}^d. \end{cases}$$

Associated to the operator $A$, is the following boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^d$:

$$A^\varepsilon u^\varepsilon(x) \equiv \frac{\partial}{\partial x_k}\left(a_{kl}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_l}\right), = f(x), x \in \Omega$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega.$$

Here $\varepsilon > 0$ is a small parameter representing the period of oscillation of the coefficients. The fundamental question is to know their effect on the solution $u^\varepsilon$. The insight of [1] is that the solution has a two-scale structure: large scale variable $x \in \Omega$, small scale variable $y = \frac{x}{\varepsilon} \in Y$ and the two scales are separated. Exploiting this feature and using the two-scale expansion, the following result is obtained:

$$u^\varepsilon(x) = u(x) + \varepsilon\chi^k\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_k}(x) + O(\varepsilon^2),$$

which shows the oscillation of the solution. Here $\chi^k(y)$ are solutions satisfying

$$A\chi^k = \frac{\partial a_{kl}}{\partial y_l}, \quad \chi^k(y) \text{ is periodic}, \quad k = 1 \dots d.$$

The function $u$ is the solution of the homogenized system:

$$Qu \equiv -\frac{\partial}{\partial x_k}\left(q_{kl}\frac{\partial u}{\partial x_l}\right) = f(x) \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

The homogenized matrix $(q_{kl})$ is obtained from $(a_{kl})$ and $(\chi^k)$ via averaging over the periodic cell:

$$q_{kl} = \left\langle a_{mn} \frac{\partial}{\partial y_n}(\chi^k + y_k) \frac{\partial}{\partial y_m}(\chi^l + y_l) \right\rangle.$$

Moreover, one proves

$$a_{kl}\left(\frac{x}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial x_l} \rightharpoonup q_{kl} \frac{\partial u}{\partial x_l}, \quad k = 1 \ldots d.$$

This is not obvious because the sequences involved are oscillating and converge only weakly. This is a typical fundamental difficulty present in all homogenization problems. Traditional approach to overcome this difficulty was to seek estimates on the second derivatives of the solution independent of $\varepsilon$ and as a consequence deduce strong convergence of its gradient. Unfortunately, such estimates involve Lipschitz norm of the coefficients which blows up as $\varepsilon \to 0$. Of course, there are estimates on the solution involving the $L^\infty$-norm of the coefficients (e.g.: $L^\infty$ estimates, de Giorgi estimates). On one hand, they are based on maximum principle, Harnack inequality and so their use is rather limited to second order equations. On the other hand, they do not imply the strong convergence of the gradient. The conclusion is that the results of two-scale expansion are new and cannot be obtained by classical means. It is important to note that $q_{kl}$ is rather a complicated average involving $a_{kl}$. This is one of the effects of oscillations in this simple situation, and this phenomenon reminds us of more complicated and mysterious eddy conductivity in fluid mechanics. The above expansion method, though not rigorous, has been developed further to handle more complicated models [2]. For a treatment of eigenvalue problems see [3,4].

## 2. *Method of oscillating test functions [5]*

This is a rigorous method to pass to the limit and describe fluctuations around it in problems which need not have periodic microstructure as in the above situation and which admits only energy type estimates and the resulting weak convergence. This is in contrast to classical methods which required strong convergence. As the name suggests, the main task is to produce suitable test functions adapted to the problem which are then used as multipliers in the problem. In the same spirit, with the choice of suitable test functions in the definition of the viscosity solution, one can also homogenize certain equations of Hamilton–Jacobi type and fully nonlinear elliptic PDE of order two [6,7].

## 3. *Method of two scale convergence [8]*

This is a new notion of convergence, finer than the classical weak convergence. Its important property is that there is a compactness result analogous to Banach–Alaoglu theorem. When one applies it to the problem under consideration, one obtains what is called two-scale homogenized system, which contains not only the homogenized limit but also fluctuations around it. This method is capable of dealing with the case where oscillations are present even to the leading order.

## 4. *Periodic unfolding method [9]*

This method is based on the introduction of the unfolding operator which associates a function of two variables (slow, fast) to each function of a single variable. Two-scale convergence is then merely weak convergence of the unfolded sequence.

## 5. *Bloch wave method [10,11]*

Inspired from ideas in solid state physics, one works in Fourier space in contrast to earlier methods. This gives rise to new type of questions and in particular new interpretation of the homogenized matrix in terms of spectral information on $A$. Along with the previous ones, this method is also restricted mainly to problems involving periodic microstructures. An important virtue of the method is its ability to make the construction of the oscillating test functions more systematic and to introduce higher order homogenized coefficients [12,13].

## 6. *Method of gamma convergence [14]*

Based on a new notion of convergence in calculus of variations, this method is suitable for treating problems of oscillating type, which admit variational structure. This is an important subclass, which includes ground state problems.

## 7. *Hybrid methods*

A combination of methods mentioned above have proved to be very effective and powerful. For wider applications, it is important to avoid using methods based on maximum principle, which is applicable only for scalar equations of order upto two. Combining two-scale convergence and Bloch waves, a new hybrid method introduced in [15] justified effective mass theorems and Anderson-type localization phenomena [16], involving periodic potentials interacting with the periodic microstructure.

## 8. *Compensated compactness method [17]*

It is well known that compactness is a basic ingredient required in any approximation. Traditional tools used in this connection are Arzela–Ascoli, Rellich type theorems which guarantee compactness provided we have better regularity and control on oscillations. Characteristic feature of homogenization problem is that we do not have such controls. Compensated compactness is somewhat a radical departure from the above traditional idea. Though regularity is still at the background, it is hidden and very subtle. This method has resolved some of the long standing problems of PDE. One crowning glory is the problem of characterizing all homogenized matrices which can be obtained by mixing two homogeneous material conductors $(\alpha, \beta)$ taken in a fixed proportion $(\gamma)$ by varying the microstructure, namely the geometry of mixing. In our notation, this amounts to taking

$$a_{kl}(y) = \alpha \chi_T(y) + \beta \chi_{Y \setminus T}(y),$$

$$\text{with} \quad |T| = \gamma |Y|, \quad 0 \leq \gamma \leq 1,$$

and the task is to characterize all possible homogenized matrices $(q_{kl})$ obtained by varying measurable subsets $T \subset Y$ [18]. Another example [19] is concerned with the large data Cauchy problem for nonlinear hyperbolic conservation laws of two equations in one space variable. Recently, same point of view is followed to construct bounded oscillating solutions demonstrating non-uniqueness of finite energy solutions to the Euler system modeling the flow of incompressible ideal fluids [20].

## 9. *Probabilistic method*

Exploiting Feynman–Kac integral representation of the solution involving non-divergence type operators, the problem is transformed to the study of behavior of probability measures on infinite dimensional function spaces. Homogenizarion process then depends on the ergodic properties of the associated diffusion process [1,21].

## 10. *H-measures [22]*

Ever since the seminal works of Hörmander on propagation of regularity, phase space methods in PDE have proved to be very effective and powerful. However, they are not applicable as such to our situation because we are dealing here with a family of problems, rather than a single one. Since oscillation and concentration of solutions of these problems are major sources of non-compactness, one seeks a phase space description of them. This is provided by H-measures, along with variants such as Wigner measures [23]. One of the achievements of H-measures is the description of propagation of energy of high frequency waves, well beyond the formation of caustics [24]. There is generalization of these measures which are found to be extremely useful. For instance, H-measures on a finite dimensional phase space taking values in an infinite dimensional target space are useful to establish compactness of averages [25], a property which is crucially needed in the study of the Boltzmann equation [26]. On the other hand, one can also introduce a version of these measures on infinite dimensional phase spaces which is an important tool in the description of mean field limit of quantum mechanics of large number of particles [27].

## 11. *Conclusion*

In this article, starting with a simple model problem, we surveyed various developments in homogenization theory. The conclusion is that we have a good understanding of problems involving two scales. Future challenge lies with problems involving large number of scales, interacting among themselves. Some examples have been treated; see [28–31]. Fluid mechanics, quantum physics and material science are full of such non-compact problems [32,33]. Future challenge lies in producing tools and concepts finer than the ones mentioned above to understand these problems.

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## IV Viscosity solutions in PDE

Different solution concepts for differential equations have evolved over the years depending on the needs. Classical and weak solutions are the first few. Viscosity solution concept, a recent entrant, has been quite successful in handling degenerate elliptic equations and passage to limits. For a weak solution to verify the equation, integration by parts is used to shift the derivatives onto the smooth test function whereas for the viscosity solution $u$, the smooth test function $\phi$ is to satisfy the equation at every maximum or minimum point of $u - \phi$ for a sub or super solution. Thus solution space could be that of continuous functions or even semi-continuous ones.

One of the main motivations for viscosity solution theory was the need in optimal control of systems governed by systems of ODEs, in particular, dynamic programming principle, to characterize the value function as the unique 'solution' of the associated partial differential equation, even when the value function lacked regularity. In [6], Crandall and P-L Lions introduced the notion of viscosity solutions for the first order PDEs of Hamilton–Jacobi type

$$u_t + H(x, u, Du) = 0$$

and proved the uniqueness of this solution, when $H$ is monotonic in the $u$ variable. Several equivalent

definitions followed, simplifying the approach. All these developments led to a spectacular simplification of the theory of deterministic differential games and also provided a sound mathematical foundation for stochastic differential games. See for example, the books [4,9] and various references there.

For the second order degenerate elliptic equation, Jensen proved (see [11]) the uniqueness of viscosity solutions and a simplified approach was given in [12]. In both first and second order equations, stability under passage to limits was one of the useful outcomes.

Extension of the theory to parabolic equations and singular equations, like mean curvature equations,

$$u_t - |Du|\mathrm{Div}\left(\frac{Du}{|Du|}\right) = 0$$

followed. This and many other equations arising from geometrical considerations, present singularities at $|Du| = 0$. This was overcome by Evans and Souganidis [8] and Chen, Giga and Goto [5], by suitably adapting the definition. In the case of first order equations, a similar first order singular Hamilton–Jacobi equation was analyzed by [10] and [15]:

$$u_t + \frac{H(x, Du)}{f(x)} = 0$$

with $f(0) = 0$.

The scope of the theory has further been extended relaxing many of the initial assumptions. If $H(x, u, Du)$ is non-increasing in $u$, in general, this problem need not admit a continuous viscosity solution. This kind of problems are studied and formula for discontinuous viscosity solution is given in [1] and [2] when $H(x, s, p) = H(s, p)$ is convex and positively homogeneous of degree one in $p$.

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## V Hyperbolic PDEs

Hyperbolic equations arise from modeling various physical phenomena which occur in gas dynamics, shallow water theory, nonlinear elasticity, magneto-fluid dynamics, etc. In their derivation the small scale effects due to viscosity, capillarity which give smoothing effects due to higher order terms are ignored. Nonexistence of smooth solutions lead to the formulation in a weak sense and for well-posedness and stability reasons additional conditions called entropy conditions are needed. A complete understanding of well-posedness and qualitative properties of solutions for initial

boundary value problems for these systems is far away. But there are great advances for hyperbolic equations in one space variables starting with the work of Hopf (1950) and Lax (1954) and Glimm (1965). The solution of Riemann problem by Lax, and existence theory of Glimm using a approximation involving Riemann problem laid the foundation for the theory of hyperbolic systems in one space variable. These results were generalized for the case with degenerate characteristic fields by many authors Liu, Dafermos, LeFloch, Tzavaras, Chen and others. Results on continuous dependence and asymptotic behavior of solutions for various situations were proved by Liu, Bressan, LeFloch. A theory of nonclassical shocks when characteristic fields are degenerate is developed by LeFloch and his collaborators. The complete proof of convergence of vanishing viscosity approximations for general hyperbolic equations in one space variable by Bressan and Bianchini (2005) is the most significant work after Glimm's method.

In India, works are carried out in this area at Tata Institute of Fundamental Research and Indian Institute of Science. Analysis of geometrical optic appxoximations for general scalar conservation laws, derivation of Lax formula for solutions of scalar convex conservation laws, with boundary conditions, construction of solutions of conservation laws with discontinuous flux, nonconservative hyperbolic equations, analysis of $\delta$ shock waves for nonstrictly hyperbolic case, analysis of boundary layers that may arise in approximate solutions of entropy weak solutions of systems of conservation laws and the local structure of the set of admissible boundary values for both the Vanishing viscosity and difference approximations, analysis of physical viscosity and capillarity effects in a system of conservation laws arising in liquid-vapour phase dynamics, for conservation laws whose flux functions are discontinuous in the space variable introducing a proper framework of entropy which provides uniqueness and also designing Godunov type numerical schemes and their study theoretically and numerically, are some of the important contributions from the Tata Institute of Fundamental Research group at Bangalore and Mumbai. The people involved in the works are Adimurthi, P S Datti, K T Joseph, Siddartha Misra, M Vanninathan and G D Veerappa Gowda.

## VI Geometrical optics and kinematical conservation laws

Asymptotic theory of geometrical optics, to discuss propagation of a wavefront, goes back to early 19th century but its application to study the propagation of a small amplitude nonlinear wave was started by J B Keller, G B Whitham, Y Choquet-Bruhat and D F Parker. Their work basically dealt with the same problem but by a very different approach: purely intuitive to a very formal one and with a very simple physical model to a quite complex one and also with a general hyperbolic system. But they had a common aim: to derive the transport equation for a small wave amplitude $w$ along a nonlinear ray which stretches due to genuine nonlinearity. Prasad derived such a transport equation, where in addition he took into account the diffraction of the nonlinear rays due to a non zero gradient of the amplitude $w$ on the nonlinear wavefront $\Omega_t$. In an isotropic medium in non-dimensional variables, his equations take the simple form:

$$\frac{dx}{dt} = (1 + kw)\mathbf{n}, \quad \frac{d\mathbf{n}}{dt} = -k\mathbf{L}w,$$

$$\frac{dw}{dt} = -\frac{1}{2}\langle \nabla, \mathbf{n} \rangle w, \tag{1a,b,c}$$

where $\mathbf{n} =$ unit normal of $\Omega_t$, $\mathbf{L} = \nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle$, $k$ is the nonlinearity constant and $\frac{d}{dt} = \frac{\partial}{\partial t} + (1 + kw)\langle \mathbf{n}, \nabla \rangle$. Unlike the previous results, the presence of the term $-k\mathbf{L}w$, gives the additional nonlinear diffraction of rays and it causes resolution (found experimentally and verified numerically) of a linear caustic and gives finite value of $w$ everywhere. This term is necessary in order that the rays given by (1a) satisfy Fermat's principle of stationary time.

The above system of equations now leads to a new phenomenon, appearance of a kink (called shock-shock by Whitham in his shock dynamics). Desire to study formation and propagation of kinks on a propagating surface $\Omega_t$ (in particular on a nonlinear wavefront), led to the discovery of purely

geometrical results: kinematical conservation laws (KCL). Morton, Prasad and Ravindran derived 2-D KCL governing the evolution of a moving curve $\Omega_t$ in a plane and Giles, Prasad and Ravindran derived 3-D KCL governing evolution of a moving surface $\Omega_t$ in 3-D space. The 3-D KCL and the ray equations (1a,b) are equivalent for a smooth $\Omega_t$. Consider two parameter family of curves $\xi_2 =$ constants and $\xi_1 =$ constant on $\Omega_t$ with associated metrics $g_1$ and $g_2$ such that these families evolve as $\Omega_t$ evolves with normal velocity $m = 1 + kw$. Then we can define a ray coordinate system $(\xi_1, \xi_2, t)$ in $(x_1, x_2, x_3)$-space such that $\frac{\partial}{\partial t}|_{(\xi_1,\xi_2)=\text{constant}} = \frac{d}{dt}$ in (1). The existence of dynamic curves $\xi_2 =$ constant and $\xi_1 =$ constant with unit tangent vectors $\mathbf{u}$ and $\mathbf{v}$ leads to the 3-D KCL:

$$(g_1\mathbf{u})_t - (m\mathbf{n})_{\xi_1} = 0, \quad (g_2\mathbf{v})_t - (m\mathbf{n})_{\xi_2} = 0 \quad \text{(2a,b)}$$

reducing the problem of solving equations (1) in 4 independent variables $(x_1, x_2, x_3, t)$ to that of solving 3-D KCL (2) in 3 independent variables $(\xi_1, \xi_2, t)$. 2-D KCL with the transport equation has found many applications (e.g. sonic boom) and applications of 3-D has just begun successfully. The KCL theory is an alternative theory to the level set theory.

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## VII One dimensional models for blow-up in 3D-Euler equations

It is well known (through the Clay prize) that the development of singular behavior in finite time for the incompressible Euler equations with smooth initial data is open. Apart from being a challenging mathematical problem it has deep connections with some fundamental questions in fluid flow. It has been conjectured that the appearance of finite time singularities is related to the onset of turbulence because singularity formation may be a mechanism of energy transfer from large to small scales. One of the most significant results that has been proved about regularity of the Euler equations is the Beale–Kato–Majda [1] theorem that says that the solution exists globally in time if and only if

$$\int\limits_0^T \max_x |\omega(x,s)| \, ds < \infty$$

for all $T > 0$. Here $\omega = \nabla \times v$ is the vorticity and the vector $v$ satisfies the Euler equations

$$\frac{Dv}{Dt} + \nabla p = 0, \quad \nabla \cdot v = 0,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$$

and $p$ is the unknown pressure. This made it very challenging to study the evolution equation for the vorticity

$$\frac{D\omega}{Dt} = \mathcal{D}(v)\omega,$$

where

$$\mathcal{D}_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$v(x) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y) \, dy.$$

The operator relating $\omega$ to $D(v)$ is a linear singular operator that commutes with translation. In one space dimension there is only one such operator which is the Hilbert transform. Inspired by this observation Constantin–Lax–Majda [2] proposed the one dimensional analogue

$$\partial_t \omega = \mathcal{H}(\omega)\omega,$$

where

$$\mathcal{H}(\omega) = \frac{1}{\pi}PV\int\limits_{-\infty}^{\infty} \frac{\omega(y)}{x-y} \, dy.$$

This equation was amenable to explicit analytical solution which showed that it blew up in finite time $T_0$. However it was shown by Schochet [3] that the viscous analogue

$$\partial_t \omega = \mathcal{H}(\omega)\omega + \epsilon \omega_{xx}$$

also blows up in finite time $T_\epsilon$ and $T_\epsilon < T_o$. This was in direct contradiction to a result of Constantin [4] which says that if the solution to the Euler equations is smooth then the solution to the slightly viscous Euler equations namely the Navier–Stokes equation is also smooth. To overcome this, Vasudeva Murthy [5] proposed an improvement for the CLM model

$$\partial_t \omega = \mathcal{H}(\omega)\omega - \epsilon \mathcal{H}\omega_x.$$

The solution to this also blows up in finite time $T_\epsilon$ but with $T_\epsilon > T_o$. A global analysis indicating precise conditions on the initial data that will lead to singular behavior was done by Wegert and Vasudeva Murthy [6]. Since then several workers like Sakajo, Cordoba, Chae and Dong have studied these kind of models; see Dong [7] for the latest and

Escudero [8] for the relevance of one dimensional models.

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