Current Trends
in Science

platinum jubilee special

Mathematical analysis in India

V S SUNDER

The Institute for Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India.
e-mail: sunder@imsc.res.in

As part of the Platinum Jubilee celebrations of the Indian Academy of Sciences, many mathematicians have been asked to write something about the areas in which there is work done by the fellowship of the academy and about the contributions to these areas by Indians. Mathematics being too vast a canvas, this writing has been broken down further into sub-disciplines. More specifically, regarding work in mathematical analysis, three people – namely Alladi Sitaram, V S Sunder and M Vanninathan – were entrusted with the task of coming up with something representative, subject to some constraints of length. This article comprises the inputs of Sitaram and Sunder, in (the comparatively close disciplines of) harmonic analysis and operator theory/algebras; whereas the input of Vanninathan, on PDEs, will appear in a separate article. Sunder gratefully acknowledges the inputs and help provided by many people, such as Gadadhar Misra, Rajarama Bhat and Kalyan Sinha.

1. Functional analysis

Research in functional analysis in India is pursued in varied forms and at various levels of seriousness. Thus you will find some Banach space theorists, some numerical functional analysts, operator theorists and operator algebraists of varying hues, ... The 'opinions' stated in this article are naturally influenced by the author's own tastes and limitations, and if some areas and/or people are not mentioned, that is partly due to limitations of space, partly to the author's own inadequacies, and any perceived slight is unintentional, and the author craves indulgence from the injured parties.

1.1 *Operator theory*

The most outstanding open problem in operator theory is *the invariant subspace problem*, which is easy to state: for every bounded operator T on a separable complex Hilbert space $\mathcal{H}$, does there exist a closed non-trivial subspace $\mathcal{M} \subseteq \mathcal{H}$ which is invariant under T, that is, $T\mathcal{M} \subseteq \mathcal{M}$. (However, there does exist an operator on the Banach space $\ell^1(\mathbb{N})$ which does not admit an invariant subspace [30].) It was shown, as early as 1954, that any non-zero compact operator admits an invariant subspace [5]. More recently, a striking result of Lomonosov [24] shows that if a bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ commutes with a non-zero compact operator on $\mathcal{H}$ then T admits a non-trivial closed invariant subspace. Since there exist [20] bounded linear operators that do not commute with any non-zero compact operators, the invariant subspace problem remains open! There is one other class, namely subnormal operators, for which the invariant subspace theorem has been proved (by Brown [11]). Later, a simple proof was given by [31]. The notes [19] provide an account of more recent developments in this area with an emphasis on the invariant subspace problem for operators on a Banach space.

The spectral theorem for normal operators provides both a complete set of unitary invariants (=invariants for unitary equivalence) and a canonical model. Beyond normal operators, there are only few instances where complete unitary invariants have been found, and canonical models constructed.

Keywords. Functional analysis; operator theory; operator algebras; quantum probability; quantum dynamical systems; subfactors; harmonic analysis.

The first such results are due to B Sz–Nagy and Foias. These involve the class of completely non-unitary (cnu) contractions [26]. They call their unitary invariant the characteristic function of the operator. The generalization of this theory to the multi-variate context has begun over the last decade [3,8,9,28,29].

Secondly, in the work of Carey and Pincus [13], they associate with a hyponormal operator T an operator-valued function called the mosaic of T. The mosaic of a pure hyponormal operator T with trace class self commutator is then shown to be a complete unitary invariant for T. They also prove the existence of a hyponormal operator with a prescribed mosaic.

Finally, the work of Cowen and Douglas [14] associates (to a certain class of operators, and more generally) to a commuting tuple $\mathbf{T}$ of operators, a holomorphic Hermitian vector bundle $E_{\mathbf{T}}$. They show that the equivalence class of the vector bundle $E_{\mathbf{T}}$ and the unitary equivalence class of the operator $\mathbf{T}$ determine each other. Thus a correspondence is set up between complex geometry and operator theory. More recently, some of this work has been recast in the language of Hilbert modules over a function algebra making it possible to ask natural questions involving submodules and quotient modules [17,16]. An identification of those homogeneous holomorphic Hermitian vector bundles over the unit disc which correspond to a Hilbert space operator has been completed [25]. For bounded symmetric domains, some partial answers are to be found in [4,10]. The question of when two operators in the Cowen–Douglas class are similar remained open until very recently. Now, it is answered completely, using the ordered K-group of the commutant algebra as an invariant [23].

The spectral theorem shows the usefulness of the L^∞ functional calculus for normal operators, or more generally, selfadjoint (closed) operator algebras. In early attempts to find a useful functional calculus, von Neumann introduced the notion of a spectral set for an operator T. A compact subset X of the complex plane is said to be a spectral set for T if X contains the spectrum σ_T of the operator T and $\|r(T)\| \leq \|r\|_{X,\infty}$ for every $r \in \mathrm{Rat}(X)$, where we write $\mathrm{Rat}(X)$ for the algebra of all rational functions with poles off X and $\|r\|_{X,\infty} = sup\{|R(z)| : z \in X\}$. Now, if X is a spectral set for the operator T, then the homomorphism $r \mapsto r(T)$ extends to the completion of $\mathrm{Rat}(X)$ with respect to $\|\cdot\|_{X,\infty}$. This enlarged functional calculus has several useful properties. For $X = \overline{\mathbb{D}}$, the closed unit disc, von Neumann established the surprising result: $\|p(T)\| \leq \|p\|_{\mathbb{D},\infty}$ for all polynomials p if and only if $\|T\| \leq 1$. Soon afterwards, B Sz–Nagy and Foias showed that if T is a contraction then the homomorphism $\varrho_T : \mathrm{Pol}(\mathbb{D}) \to \mathcal{L}(\mathcal{H})$ defined by $p \mapsto p(T)$ dilates. In other words, there exists a $*$-homomorphism $\tilde{\varrho}_T : C(\mathbb{T}) \to \mathcal{L}(\mathcal{K})$ of the continuous functions $C(\mathbb{T})$ into the algebra $\mathcal{L}(\mathcal{K})$ of bounded operators on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $P_{\mathcal{H}}\tilde{\varrho}_T(p) = \varrho_T(p)$ for any polynomial p, as long as $\|T\| \leq 1$ (here $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$). The von Neumann inequality follows from this dilation theorem for contractive operators T. If X is a spectral set for the operator T, does it necessarily dilate? An affirmative answer was given for 1-connected compact sets X in [1]. Although, it was widely believed that the answer would be negative in general, examples were found only very recently [18]. However, in the early seventies, Arveson showed that if one assumed X to be a 'complete spectral set' for the operator T then the induced homomorphism admits a dilation [7]. However, corresponding questions involving commuting tuples of operators appear to be very mysterious (cf. [2,17]). In his famous list of 'ten problems' [21], Halmos notes that if an operator T is similar to a contraction ($T = LSL^{-1}$ for some invertible operator L and some contractive operator S), then $\|p(T)\| \leq K\|p\|_{\mathbb{D},\infty}$ for all polynomials p. He then asks if the converse is true. Recently, Pisier has found a counter example. This example along with lot more on similarity questions from several different areas of mathematics are discussed in [27].

There is another way in which one may go beyond the spectral theorem, and that is to work modulo compact operators. The rationale is that the compact operators are limits of finite dimensional operators and therefore don't count in an essential way. Thus the two operators S, T are said to be essentially unitarily equivalent if $USU^* = T + K$ for some unitary operator U and a compact operator K. Similarly, the operator N is said to be essentially normal if $NN^* - N^*N = K$ for some compact operator K. An operator T has an essential spectrum $\sigma_e(T)$, which is the set of complex numbers λ such that $T - \lambda I$ is not invertible modulo the compacts. The index is a map $\mathrm{ind}_T : \mathbb{C} \setminus \sigma_T \to \mathbb{Z}$, which is defined naturally as the difference $\dim\ker(T-\lambda) - \dim\ker(T-\lambda)^*$. In the early seventies, Brown, Douglas and Fillmore (cf. [15]) proved that two essentially normal operators S and T are essentially unitarily equivalent if and only if $\sigma_e(S) = \sigma_e(T)$ and $\mathrm{ind}_S = \mathrm{ind}_T$. The proof of this theorem involves calculating the 'Ext' group for the algebra $C(X)$ of continuous functions on a subset X of the complex plane $\mathbb{C}$. As they have shown, the context for this calculation is algebraic topology. In particular, they show $\mathrm{Ext}(X) = \mathrm{Hom}(\pi^1(X), \mathbb{Z})$ for any subset of X of the complex plane $\mathbb{C}$. This has led to unexpected connections with K-theory and to the Atiyah–Singer index theorem. The excitement is far from over [22].

References

[1] Agler J 1985 Rational dilation on an annulus; *Ann. Math.* **121** 537–563.

[2] Jim Agler and John McCarthy E 2002 Pick interpolation and Hilbert function spaces; *Graduate Studies Math.* **44** AMS.

[3] Arazy J and Englis M 2003 Analytic models for commuting operator tuples on bounded symmetric domains; *Trans. Amer. Math. Soc.* **355** 837–864.

[4] Arazy J and Zhang G 2003 Homogeneous multiplication operators on bounded symmetric domains, preprint; *J. Funct. Anal.* **202** 44–66.

[5] Aronszajn N and Smith K T 1954 Invariant subspaces of completely continuous operators; *Ann. Math.* **60** 345–350.

[6] Arveson W 1969 Subalgebras of C*-algebras; *Acta. Math.* **123** 141–224.

[7] Arveson W 1972 Subalgebras of C*-algebras-II; *Acta. Math.* **128** 271–308.

[8] Arveson W 1998 Subalgebras of C*-algebras-III; *Acta. Math.* **181** 159–228.

[9] Athavale A 1987 Holomorphic kernels and commuting operators; *Trans. Amer. Math. Soc.* **304** 101–110.

[10] Bagchi B and Misra G 1996 Homogeneous operator tuples on twisted Bergman spaces; *J. Funct. Anal.* **136** 171–213.

[11] Brown S 1978 Some invariant subspaces for subnormal operators; *Int. Eqns. and Oper. Th.* **1** 310–333.

[12] Bhattacharyya T, Eschmeier J and Sarkar J 2005 Characteristic function of a pure commuting contractive tuple; *Int. Eqns. Oper. Th.* **53** 23–32.

[13] Carey R W and Pincus J D 1977 Mosaics, principal functions, and mean motion in von Neumann algebras; *Acta Math.* **138** 153–218.

[14] Cowen M and Douglas R G 1978 Complex geometry and operator theory; *Acta Math.* **141** 187–261.

[15] Douglas R G 1995 *C*-algebra extensions and K-homology*, (AM-95), Princeton University Press.

[16] Douglas R G and Misra G 2008 Equivalence of quotient modules-II; *Trans. Amer. Math. Soc.* **360** 2229–2264.

[17] Douglas R G and Paulsen V I 1989 *Hilbert modules over function algebras*; Pitman Res. Notes Math. **217** Longman.

[18] Dritschel M A and McCullough S 2005 The failure of rational dilation on a triply connected domain; *J. Amer. Math. Soc.* **18** 873–918.

[19] Godefroy G Some linear operators and their invariant subsets, to appear; *Lecture Note Ser. Math.* Ramanujan Math. Soc.

[20] Hadwin D, Nordgren E, Radjavi H and Rosenthal P 1980 An operator not satisfying Lomonosov's hypothesis; *J. Funct. Anal.* **38** 410–415.

[21] Halmos P R 1970 Ten problems in Hilbert space; *Bull. Amer. Math. Soc.* **76** 887–933.

[22] Higson N and Roe J 2000 *Analytic K-homology*, Oxford Mathematical Monographs.

[23] Jianga C, Guo X and Ji K 2005 K-group and similarity classification of operators; *J. Func. Anal.* **225** 167–192.

[24] Lomonosov V I 1973 Invariant subspaces for operators commuting with compact operators; *Funct. Anal. Appl.* **7** 213–214.

[25] Koranyi A and Misra G Classification of homogeneous operators in the Cowen–Douglas class, preprint.

[26] Sz–Nagy B and Foias C 1970 *Harmonic analysis of operators on Hilbert space*, North Holland.

[27] Pisier G 2001 *Similarity problems and completely bounded maps – second expanded edition*, LNMS 1618.

[28] Popescu G 1989 Isometric dilations for infinite sequences of noncommuting operators; *Trans. Amer. Math. Soc.* **316** 523–536.

[29] Popescu G 1998 Noncommutative joint dilations and free product operator algebras; *Pac. J. Math.* **186** 11–140.

[30] Read C J 1984 A solution to the invariant subspace problem; *Bull. London Math. Soc.* **16** 337–401.

[31] Thomson J 1986 Invariant subspaces for algebras of subnormal operators; *Proc. Amer. Math. Soc.* **96** 462–464.

---

2. Operator algebras

One area of Indian work in functional analysis that has received some notice from the world at large is the various forms of operator algebras, and we now focus on this. There are four branches of operator algebras where there has been some serious work done in India; they may be broadly described as quantum probability, non-commutative geometry, quantum dynamical systems and subfactors. In what follows, a paragraph or two will be devoted to describing the concerned area, and some of the Indian contributions to that area.

2.1 *Quantum probability*

Quantum probability is a generalization of the usual (classical) probability theory. Here the random variables are replaced by observables and they are self-adjoint operators on Hilbert spaces or even more generally simply self-adjoint elements of a $*$-algebra equipped with a state (evaluation of the state on a self-adjoint element gives its expectation). The crucial point is that unlike two random variables of classical probability two observables need not commute. So it is a 'non-commutative' probability theory. The

Indian school of quantum probability is led by K R Parthasarathy and K B Sinha.

K R Parthasarathy and K Schmidt [9] found close relationships between infinitely divisible distributions of Levy and Khinchin and the projective unitary Weyl representation of the Euclidean group of a Hilbert space in Boson (symmetric) Fock space. This observation and Ito's theory of stochastic differential equations motivated Hudson and Parthasarathy to initiate a theory of quantum stochastic integration [10]. This has developed into a vast theory called 'quantum stochastic calculus' and it has connections with classical probability theory, mathematical physics, quantum information theory and other fields of mathematics. In this theory (see [2]), the Boson Fock space $\Gamma(L^2(\mathbb{R}^+, \mathcal{K}))$, its continuous tensor product decomposition, and its filtration $\{\mathcal{B}(\Gamma(L^2([0,t], \mathcal{K}))) \otimes id_{\Gamma(L^2([t,\infty),\mathcal{K}))} : t \geq 0\}$ (where $\mathcal{K}$ is a separable Hilbert space) play a central role. Quantum stochastic integrators are constructed using creation, conservation and annihilation operators. This finally leads to a 'quantum Ito formula' and one is able to arrive at canonical anti-commutation relations (CAR) starting with canonical commutation relations (CCR).

The quantum stochastic analogue (called the Hudson–Parthasarathy equation) of the (deterministic) Schrodinger equation, and the counterpart of the Heisenberg equation (called the Evans–Hudson equation) describe the evolution in their respective contexts. In quantum stochastic calculus, there is the curious spectacle of both Brownian motion and Poisson processes being adapted with respect to the filtration described earlier and the possibility of describing a classical Markov chain as a quantum stochastic flow over the commutative algebra of functions on the discrete state space ([2]). Furthermore, the concept of a stop time was generalized ([3]) to the non-commutative set-up by defining it as a spectral family adapted to the filtration described earlier in the symmetric Fock space. For example, one can conceive of stopping a Brownian motion by a Poisson stop time.

2.2 *Non-commutative geometry*

The theory of operator algebras led Connes (see [4]) to a whole new way of looking at spaces and functions on them. Taking a cue from the fact that many topological and geometric properties of a classical space can be encoded into specific properties of the (commutative) algebra of (continuous or measurable) functions on the space, one studies just these properties on C* or von Neumann algebras (see e.g. [1],[4], and [5]). Intuition from quantum mechanics plays an important role here and the models are either classical spaces or their deformations.

On the other hand, quantum groups provide a big class of non-commutative spaces that are quite far removed from classical spaces. Although geometry and groups often go hand in hand in the classical context, the relationship between the geometry (as described operator-theoretically by the spectral triple of Connes) and quantum groups was not initially well understood. Later, Chakraborty and Pal showed (in [6] and [7]) that quantum groups and their homogeneous spaces fit very well in the scheme by first constructing a Dirac operator for quantum $SU(2)$ that has a non-trivial K-homology class, and is of metric dimension 3; in fact this construction was later used by Connes to prove a local index formula. These methods of Chakraborty and Pal also enable the computation of certain invariants for a large class of quantum groups and their homogeneous spaces.

Goswami ([8]) gave a definition of a 'smooth and isometric' action by a compact quantum group on a non-commutative manifold, extending similar notions in the classical case, in such a way that isometric actions automatically commute with the 'Laplacian' associated with the relevant spectral triple. In the category of quantum families of smooth isometries, the universal object can be identified with the underlying C*-algebra of the quantum isometry group, equipped with a coproduct to turn it into a compact quantum group. Goswami, with his students, has also proved that the quantum isometry group of an isospectral deformation of classical or non-commutative manifold is a suitable deformation of the quantum isometry group of the original manifold.

References

[1] Sinha K B and Goswami D 2007 *Quantum stochastic processes and non-commutative geometry* (UK: Cambridge University Press).

[2] Parthasarathy K R 1992 *An introduction to quantum stochastic calculus* (Basel: Birkhauser Verlag).

[3] Parthasarathy K R and Sinha K B 1987 Stop times in Fock space stochastic calculus; *Probability theory and related fields*; Springer Verlag **75** 317–349.

[4] Connes A 1994 *Non-commutative geometry* Acad. Press.

[5] Chakraborty P S and Sinha K B 2003 Geometry on quantum Heisenberg manifold; *J. Func. Analysis* **203** 425–452.

[6] Chakraborty P S and Arup Pal 2003 Equivariant spectral triples on the quantum $SU(2)$ group; *K-Theor.* **28(2)** 107–126.

[7] Chakraborty P S and Arup Pal 2003 Spectral triples and associated Connes-de Rham complex for the

quantum $SU(2)$ and the quantum sphere; *Comm. Math. Phys.* **240(3)** 447–456.

[8] Goswami D Quantum group of isometries in classical and noncommutative geometry; to appear in *Comm. Math. Phys.*

[9] Parthasarathy K R and Schmidt K 1972 *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*; Lecture Notes in Math. **272** (Berlin-New York: Springer Verlag) vi+107 pp.

[10] Hudson R L and Parthasarathy K R 1984 Quantum Ito's formula and stochastic evolutions; *Comm. Math. Phys.* **93(3)** 301–323.

3. Quantum dynamical systems

Quantum dynamical semigroups (QDS) are one parameter semigroups of contractive completely positive maps. They are used to describe quantum open systems, that is, those systems which have some dissipation caused due to interaction with some external system. From the physics point of view, the question addressed is: does there exist a suitable probabilistic model for the environment and its interaction with the quantum system such that the expectation of the total evolution with respect to the probabilistic variables gives back the QDS one started with. Mathematically this is part of dilation theory (the particular semigroup on the larger algebra is called the dilation of the original QDS). Ideally, both from a physical and mathematical point of view, one would like to have a semigroup of automorphisms as dilation. But unfortunately, in general for some purely mathematical reasons this is not feasible.

Building on his earlier work with K R Parthasarathy on 'weak Markov dilations', B V R Bhat [3] showed that dilation is always possible and is essentially unique if instead of semigroups of automorphisms one is happy with semigroups of endomorphisms. Semigroups of automorphisms of the algebra of all bounded operators on a Hilbert space are well-understood, thanks to some classical theorems of Wigner and Stone. However, semigroups of endomorphisms (E-semigroups) exhibit much more intricate structure. Their study was initiated by R T Powers [12]. W Arveson showed [1] a way of associating tensor product systems of Hilbert spaces with E-semigroups. These product systems are classified into three types. Type I systems are fairly well-understood as they come from familiar symmetric/anti-symmetric Fock spaces. Type II and III systems are more exotic. B Tsirelson constructed [13,14] some type III systems through exponentiation of sum systems, defined through Gaussian measures. B V R Bhat and R Srinivasan generalized [6] a famous theorem of Shale, and managed to show that the construction of Tsirelson can be understood through purely functional analytic tools, and that, in the presence of a certain additional technical condition, only type I and III examples can be generated from sum systems. More recently, M Izumi has shown that this condition is always satisfied. In the more general context of QDS on C* or von Neumann algebras, one needs to study product systems of Hilbert C* (or von Neumann) modules. Such a theory has been developed by Bhat, Skeide and their co-authors [2,5].

The dilation theory of QDS is intimately connected with quantum stochastic calculus [11]. In fact the pioneering work of R L Hudson and K R Parthasarathy [10] showed that dilation to type I E-semigroups (second quantization of shift on the Fock space) is possible by solving quantum stochastic differential equations, if the semigroup is uniformly continuous. A Mohari and K B Sinha [7] extended this theory to the case of infinite degrees of freedom. D Goswami, A Pal and K B Sinha [8] could extend the theory further to more general C*-algberas. Bhat [4] gets a stronger bridge between dilation theory and quantum stochastic calculus by showing that dilation obtained through Hudson–Parthasarathy theory is the unique minimal dilation under suitable choice of structure coefficients. If the QDS is merely strongly or w*-continuous, in full generality we do not know the structure of the generator and also the dilation problem is complicated. If it is assumed, further, that there is an action of a Lie group on the algebra which leaves invariant a semifinite faithful trace on it, and that the QDS is symmetric and commutes with the group action, then there is a satisfactory solution in [9].

References

[1] Arveson W 1989 Continuous analogues of Fock spaces; *Mem. Am. Math. Soc.* **80(409)** 1–66.

[2] MR2065240 (2005d:46147) Barreto Stephen D, Bhat B V Rajarama, Liebscher Volkmar and Skeide Michael 2004 Type I product systems of Hilbert modules; *J. Funct. Anal.* **212(1)** 121–181.

[3] MR1329528 (96g:46059) Bhat B V Rajarama 1996 An index theory for quantum dynamical semigroups; *Trans. Amer. Math. Soc.* **348(2)** 561–583.

[4] MR1804156 (2002e:46083) Bhat B V Rajarama 2001 Cocycles of CCR flows; *Mem. Am. Math. Soc.* **149(709)** x+114 pp.

[5] MR1805844 (2001m:46149) Bhat B V Rajarama and Skeide Michael 2000 Tensor product systems of Hilbert modules and dilations of completely positive semigroups; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3(4)** 519–575.

[6] MR2126876 (2006e:46075) Bhat B V Rajarama and Srinivasan R 2005 On product systems arising from sum systems; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8(1)** 1–31.

[7] MR1176275 (93i:81112) Mohari A and Sinha Kalyan B 1990 Quantum stochastic flows with infinite degrees of freedom and countable state Markov processes; *Sankhy a Ser. A* **52(1)** 43–57.

[8] MR1811165 (2002f:81057) Goswami Debashish, Pal Arup Kumar and Sinha B Kalyan 2000 Stochastic dilation of a quantum dynamical semigroup on a separable unital C^* algebra; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3(1)** 177–184.

[9] MR2299106 (2008e:81070) Goswami D and Sinha K B 2007 *Quantum stochastic processes and noncommutative geometry* (UK: Camb. Univ. Press).

[10] MR0745686 (86e:46057) Hudson R L and Parthasarathy K R 1984 Quantum Ito's formula and stochastic evolutions; *Comm. Math. Phys.* **93(3)** 301–323.

[11] MR1164866 (93g:81062) Parthasarathy K R 1992 *An introduction to quantum stochastic calculus* Monographs in Mathematics **85** (Basel: Birkhauser Verlag) xii+290 pp., ISBN: 3-7643-2697-2.

[12] Powers Robert T 1988 An index theory for semigroups of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ and type II_1 factors; *Canad. J. Math.* **40(1)** 86–114.

[13] Tsirelson B 2000 From Random sets to continuous tensor products: answer to three questions of W Arveson, preprint math/0001070.

[14] Tsirelson B From slightly coloured noises to unitless product systems, preprint math. FA/0006165.

---

4. Subfactors

The theory of von Neumann algebras was shown by its founding fathers, F J Murray and John von Neumann, to have the so-called factors (= von Neumann algebras with trivial centers) as its 'building blocks'. They initially classified these factors into three broad types. The type I factors were familiar and the analogues of matrix algebras. The type III factors were later shown – as a consequence of the celebrated Tomita–Takesaki theorem – to be related to 'the ergodic theory on type II factors and their automorphisms. Type II factors came in two flavours – the finite or type II_1, and the infinite or type II_∞ (which itself is an infinite 'amplification' of a II_1 factor). Thus II_1 factors occupy centrestage.

A II_1 factor M shares many similarities with matrix algebras. Thus, they admit a faithful tracial state, i.e., a linear functional $\mathrm{tr} : M \to \mathbb{C}$ satisfying

1. $\mathrm{tr}(xy) = \mathrm{tr}(yx)\ \forall x, y \in M$
2. $\mathrm{tr}(1) = 1$
3. $\mathrm{tr}(x^*x) > 0$ for all $0 \neq x \in M$

Further such a trace is unique. The next point of similarity – as well as difference – lies in the fact that an M-module $\mathcal{H}$ is classified up to (unitary) equivalence by a number, usually denoted $\dim_M(\mathcal{H})$; the difference is that for II_1 factors, this dimension can be any *real* number in $[0, \infty]$. This continuously varying dimensions got von Neumann very excited and thinking about continuous geometries.

Now we move the camera forward from the late 1930s to the mid 1980s, when Vaughan Jones started to look at subfactors, i.e., a pair (or a unital inclusion, to be precise) $N \subset M$ of II_1 factors. He observed that in such a case, the ratio $(\dim_N(\mathcal{H}))/(\dim_M(\mathcal{H}))$ was independent of the choice of an M-module $\mathcal{H}$ with $\dim_M(\mathcal{H}) < \infty$, and called this ratio the index of the subfactor and denoted it by $[M : N]$. Going by past experience with II_1 factors, one might expect to be able to find subfactors of all possible index values, but Jones discovered the intriguing fact that this 'effect could not be turned on till after 4!' More precisely, he showed – in [1] – that the set of possible index values for subfactors was the set $\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, \infty)$. Among subfactors, the so-called hyperfinite irreducible ones were somehow special, and it is still not entirely clear what possible index values could be attained by such subfactors. Some series of examples of such subfactors were obtained and some of the index values so attained – for example $(\frac{n+\sqrt{n^2+4}}{2})^2, n \geq 1$ – were obtained in [6]. (The above expression, for $n = 1$, is the square of the golden mean and had been shown by Jones to be the smallest non-integral index value!)

It was proved by Popa – see [7] – that a certain class of 'good subfactors' were determined, up to isomorphism, by a certain standard invariant which was later recast by Jones – see [2,3] – into a diagrammatic (as against a combinatorial) structure called planar algebras. Some interesting results about planar algebras have also been subsequently obtained by Sunder and Kodiyalam – see [5] or [4] for instance.

References

[1] Jones V F R 1983 Index for subfactors; *Inventiones Mathematicae* **72** 1–15.

[2] Jones V F R *Planar Algebras I*, New Zealand J. Math., to appear, and arXiv.math.QA/9909027.

[3] Jones V and Sunder V S *Introduction to subfactors.*

[4] Kodiyalam V, Landau Z and Sunder V S 2003 The planar algebra associated to a Kac algebra; *Proc. Indian Acad. Sci.* (Math. Sci.) **113** 15–51.

[5] Kodiyalam V and Sunder V S 2004 A complete family of numerical invariants for a subfactor; *J. Funct. Anal.* **212** 1–27.

[6] Sunder V S 1992 *From hypergroups to subfactors, operator algebras and operator theory* (ed.) Arveson W *et al* Proc. OATE 2 Conf. Romania 1989, Pitman Research Notes in Mathematics (Essex, UK: Longman Scientific) 198–216.

[7] Popa S 1991 *Subfactors and classification in von Neumann algebras*; Proc. Int. Congress Math. Kyoto 1990 987–996.

5. Harmonic analysis

This is a thumbnail sketch of some of the work in harmonic analysis in India over the past twenty years, with due apology to the Indian harmonic analysis community that many topics which are pursued actively in India have not been included.

5.1 *Harmonic analysis on* $\mathbb{R}^n$

Some harmonic analysts in India have been active in modern Euclidean fourier analysis (or what one might call 'hard analysis'), thus shifting the focus away from classical Fourier series. These include investigations on singular integral operators, Carleson measures, transference and multipliers. An excellent reference to this area is the encyclopedic treatise of Stein [4]. A few people trained in this area have also been working on problems in the theories of wavelets, special functions and integral geometry. Roughly speaking, integral geometry deals with the recovery of functions (and more generally measures) when one has information about the integrals of these functions on a limited class of sets. These questions are in turn related to theorems in spectral analysis.

5.2 *Harmonic analysis on Lie groups*

In what follows, by 'harmonic analysis on Lie groups', we mean the body of theorems which are analogues of results of classical harmonic analysis, in the context of Lie groups. Thus, we have not said anything about the impressive results obtained in India in the representation theory of Lie groups per se.

After Harish-Chandra had completed much of his monumental work on representation theory of non-compact semi simple Lie groups [6], the stage was set for working out detailed harmonic analysis on this class of groups and symmetric spaces associated with these groups [3,2].

In India, considerable work has gone on in recent years in these areas. One example of this is the work involving uncertainty principles in harmonic analysis on Lie groups and generalizations of the theorems of Hardy and Beurling and their connections with the heat kernel. (Roughly speaking, uncertainty principles in classical Fourier analysis say that a function and its Fourier transform cannot be 'concentrated' or 'localised' simultaneously. Depending on how one defines these terms, a host of uncertainty principles are obtained.) An overview of uncertainty principles in harmonic analysis in general can be found in [1] and in harmonic analysis on Lie groups in [5]. Recently a good deal of attention has been paid to harmonic analysis of general L^1 and L^p functions on symmetric spaces (and AN-groups) without any assumptions of K-finiteness. Here one is interested, in the set-up of symmetric spaces and AN-groups, in results such as the Hausdorff–Young inequality, Hardy–Littlewood–Paley inequalities and modified versions of the Wiener–Tauberian theorem. Applications of harmonic analysis on symmetric spaces to integral geometry have also been taken up by Indian mathematicians. A good deal of work has been done on nilpotent Lie groups, in particular, the Heisenberg group. There has also been some recent activity in the emerging area of harmonic analysis on crowns.

References

[1] Folland G B and Sitaram A 1997 The uncertainty principle: a mathematical survey; *J. Fourier Anal. Appl.* **3** 207–238.

[2] Gangolli R and Varadarajan V S 1988 *Harmonic analysis of spherical functions on real reductive groups*, Springer Verlag.

[3] Helgason S 1994 *Geometric analysis on symmetric spaces*, AMS.

[4] Stein E M 1993 *Harmonic Analysis*, Princeton University Press.

[5] Thangavelu S 2004 *An introduction to the uncertainty principle*, Birkhauser.

[6] Warner G 1972 *Harmonic analysis on semisimple Lie groups*, vols I & II, Springer Verlag.