Some New Trends in Differential Equations

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5th April 2008

Overview

Mathematical Models

- Model :
  Any natural process, physical phenomenon : Melting of ice, Radioactive decay , Spring motion, heat conduction $\cdots$
- Mathematical model :
  A relation connecting various quantities involved in the process using a basic principle or law
- Motivation :
  To understand the phenomenon ; To make predictions ; to control by using various parameters; $\cdots$
- Tools of Study :
  Analysis of the equation

Brief History

- How to model change?
  Use : $\frac{d}{dt}$
- Tool for the study of change in real world :
  Differential Equations
- Newton introduced derivatives and integrals to understand physical phenomena
- Newton's Equations of Motion . 17th century .
- ODE's and First Order PDE's
- Hamilton's system from Mechanics,
  Hamilton-Jacobi Equation - Optics and Mechanics

Brief History

- Vibrating String - D'Alembert $\sim$ 1752 ,
  One Dimensional wave equation.
- Vibrating Membrane and Vibrating Elastic Solids - Euler and Bernoulli
  Higher Dimensional wave equation ($n = 2$ and 3)
- Gravitational potential fields - Laplace $\sim$ 1780
- Analytic Theory of Heat Conduction - Fourier $\sim$ 1815
- Equations of mathematical physics, Existence, Uniqueness, regularity - last century
- Control probelems - last few decades
- Models from Biology and Finance - recent years

Main Questions

- Do these differential equations admit a solution?
- Is that solution unique?
- Where does this solution exist?
- How does this solution behave : bounded? decays to zero?
- Is it stable ?
- Can we control the behaviour of these solutions?

A Model

- Optimal flight trajectory:
  How to choose a rocket trajectory as close to the desired flight path with minimal fuel consumption over a time interval.?
- $x(t)$ is the deviation of the rocket trajectory.
- $\dot{x}(t) = A(t)x(t) + B(t)u(t), x(0) = x_0.$
- A, B are matrix valued functions.
- The control $u(t)$ is the rocket thrust.
- x_0 is the initial condition.
- Fuel consumption cost :

$$\mathcal{J} = \int_0^1 l(t, x(t), u(t))dt.$$

Optimal Control

- **Qn :** Can we minimize the Cost functional $\mathcal{J}$ over admissible controls ?
- Infinite Horizon problem: $t \in [0, \infty)$;

$$\mathcal{J} = \int_0^\infty l(t, x(t), u(t)) e^{-\lambda t} dt$$

λ discount factor

- Finite Horizon problem: $t \in [0, T]$
- **Qn :** Does the optimal control exist ? Can we characterize it? Can we compute it?

Dynamic Programming

- Idea: Consider Value function

$$V(x) = \min \mathcal{J}(x, u)$$

  with different initial points x and use this function to get the optimal control.
  Works well when V is smooth.

- For the infinite horizon problem, if $V(x)$ is differentiable at x, then it satisfies the HJB equation:

$$\lambda V(x) + \sup_{\alpha}\{- < f(x, \alpha), DV(x) > - l(x, \alpha)\} = 0$$

- For the finite horizon problem, value function $V(x, t)$ satisfies time dependent HJB equation.

Pontryagin Maximum Principle

For finite dimensional optimization problems with constraints, Lagrange multipliers help to obtain necessary conditions for optimality.
This idea can be extended to optimal control too.

Theorem

Under usual smoothness assumptions, let $x^*(\cdot), u^*(\cdot)$ be optimal process. Then there exist $\lambda_0 \geq 0$, $\lambda \geq 0$, $\mu \in \mathbb{R}^d$ and a vector function $p(\cdot)$ satisfying the adjoint equation together with boundary conditions given by tranversality relations, such that a.e., $t \in [0, T]$,

$$\min_u H(t, x^*(t), p(t), u) = H(t, x^*(t), p(t), u^*(t)).$$

Simple Example

- **Qn** Can we suitable steer the car so as to reach B starting from A?
- Model :

$$x''(t) \ = \ u(t) \ ; \ x(0) = x_0, \ x_1 = x_1$$

- Control u is acceleration

Kalman's Rank Condition

- More generally

$$X'(t) \ = \ AX(t) + Bu(t)\ ;\ X(0) \ = \ X_0,$$

X is n vector; A is nxn matrix; B is nxm matrix

- **Qn :** When is the system A, B controllable ?
- If and only if the matrix $[B, AB, A^2B, \cdots, A^{n-1}B]$ is of full rank

Example

- A mass-spring system :

$$x''(t) + x(t) = 0$$

- $x(t)$ is the displacement from the equilibrium position
- An accelerometer attached to the mass to measure the acceleration
- **Qn** Is this observation enough to reconstruct the solution ?
- Related to controllability question for the adjoint system

Example

- Inverted pendulum is unstable
- **Qn :** Can we find a control to stabilize it ?
- Mathematical model : $\theta(t)'' - \theta(t) = u(t)$
- More generally if the system $X'(t) = AX(t)$ is unstable, can we choose $Bu(t)$ so that the resulting system

$$X'(t) = AX(t) + Bu(t)$$

is stable?

Feedback Control

- If we use feedback control $u(t) = KX(t)$, then

$$X'(t) \;=\; (A + BK)X(t)$$

  is controllable under some conditions
- Choose K so that the eigenvalues of the matrix $A + BK$ are negative.
- For linear constant coefficient system, controllability implies stabilizability

Bibliography

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