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Stochastic Oscillations of General Relativistic Disks Described by a Fractional Langevin Equation with Fractional Gaussian Noise

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Abstract:	A generalized Langevin equation driven by fractional Brownian motion is used to describe the vertical oscillations of general relativistic disks, By means of numerical calculation method, the displacements, velocities and luminosities of oscillating disks are explicitly obtained for different Hurst exponent H. The results show that as the H increase, the energies and luminosities of oscillating disk are enhanced, and the spectral slope at high frequencies of the power spectrum density of disk luminosity is also increased, it could explain the observational features related to the IntraDay Variability of the BL Lac objects.
Response to Reviewers:	Responses to the reviewers' Comments Dear Editor and Reviewers We would like to express our sincere thanks for your suggestions and help to my work. We have changed our manuscript according to the reviewers' comments, and list the responses to them as following. 1) In the Introduction Section the authors mention that the IDV can be related to the oscillations of accretion disks. Other theoretical explanations these processes have also been proposed, and the authors must discuss them briefly. Reply 1：According to the referee's suggestion we have mentioned some theoretical explanations for IDV in the Introduction Section. Such as the models based on spots, flares, and spiral shocks in accretion disks, Doppler boosted relativistic jet model, interstellar scintillation model, and deterministic non-linear system model, etc. The concerned references are also added. 2) A reference must be given for Eq. (1), which was already proposed in the literature. Reply 2: The Eq.(1) was first derived in detail by Harko et al. (2012). We have added the citing of this literature for it. 3) The authors should include, in an Appendix, for example, a brief discussion of the

fractional calculus and of the definition of the Caputo fractional derivative.

Reply 3: According to the referee’s suggestion we have added an Appendix, where we present the several definitions of fractional derivatives, and obtain the differential approximation for Caputo derivative.

4) A brief description of the numerical algorithm used to integrate the fractional differential equation is also necessary to be presented.

Reply 4: in the revised version, we added a Section to introduce the numerical approach to the fractional Langevin equation (shown in Section 3).

5) Can values of $\alpha $ higher than 2 (of the order of 2.5) be obtained in the present model?

Reply5: According to the fitting results by our modes, the values of α can be higher than 2.When the accretion disks are subjected to a fixed damping and white Gaussian noise, Harko and Mocanu (2012) have demonstrated the range for the spectral slope of their oscillating luminosity is from 1.35 to 1.65 because of the different gravitation radius and spin parameter. It is roughly the same with the result computed by our present model in the case of H tending to 1/2 as shown in Fig.4(a). And we can find that as the H exponent increases, the order (2-2H) of the fractional displacement derivative is decreased, but the value of α is increased. When $H=0.9, \alpha=2.0397$.

6) There are a large number of typing mistakes and incorrect English formulations in the manuscript. The author must carefully check and correct the text. For example, "fraction" appearing in many places of the text must be changed to "fractional". In the text below Fig. 3 "illuminate" must be replaced by other word, while "varieties" must be changed to variations.

Reply 6：We have checked the English, and found some mistakes, and corrected them in the revised version.

Stochastic Oscillations of General Relativistic Disks Described by a Fractional Langevin Equation with Fractional Gaussian Noise

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Abstract. A generalized Langevin equation driven by fractional Brownian motion is used to describe the vertical oscillations of general relativistic disks, By means of numerical calculation method, the displacements, velocities and luminosities of oscillating disks are explicitly obtained for different Hurst exponent *H*. The results show that as the *H* increase, the energies and luminosities of oscillating disk are enhanced, and the spectral slope at high frequencies of the power spectrum density of disk luminosity is also increased, it could explain the observational features related to the IntraDay Variability of the BL Lac objects.

Key words: accretion disk, power spectrum density, generalized Langevin equation, fractional Brownian motion

1 Introduction

It is well known that the IntraDay Variability (IDV) phenomena of compact, radio-loud Active Galactic Nuclei (AGN) have been widely observed in different bands (Wagner & Witzel 1995), and IDV measurements of BL Lac objects play an essential role as a potentially tool for probing the physics and geometry in the innermost part of the AGN (Krichbaum et al. 2002). However, the origin of this variability is not yet clear, so a number of theoretical models have been proposed to explain this observed phenomenon. For example, Abramowicz et al (1991), Mangalam & Wiita (1993), and Chakrabarti & Wiita (1993) have suggested the models based on spots, flares, and spiral shocks in accretion disks as a contributor to variability in AGN. Blandford & Königle (1979) have proposed that the IDV of radio emission is originated from a Doppler boosted relativistic jet. Rickett et al (1990,2001) , and Savolainen et al (2008) have demonstrated that the interstellar scintillation could successfully account for the IDV in some sources. Karak et al (2010), and Mineshige et al (1994) have suggested that the IDV is a deterministic non-linear system (e.g. chaotic, self-organized structure). However, there has been no consensus on the physical nature of IDV.

In the variability of AGN with supermassive black hole, IDV occurs in the optical domain of output luminosity with Power Spectral Distribution (PSD) for which the spectral slope α is neither 0 nor -2 (Carini et al. 2011; Remillard & Mcclintoc 2006), and it is implied that the IDV behaviour may be described by a chaotic system with some additional noise. So it was suggested that the IDV of AGN can be related to the stochastic oscillations of accretion disks. For example, Harko et al. (2012,2014) and Leung et al. (2011,2014) have considered a generalized Langevin equation with white or color noise to describe the vertical oscillations of accretion disks, and have used the luminosities of stochastically perturbed disks in both the static Schwarzschild and rotating Kerr geometries to well explain the IDV of AGN. Based on their model, Wang et al. (2013,2015) have investigated the luminosities and PSDs both in the non-relativistic and relativistic cases， and discussed the stochastic resonance phenomenon in PSD curves.

In recent years, evolution equations driven by fractional Brown motion (FBM) have been attracted the attentions of many researchers due to lots of application to problems in viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, etc (Huang & Liu 2005; Liu et al. 2003). FBM first was introduced in dynamics equations of accretion disks in MHD framework by Mocanu et al. (2014), they considered an equilibrium rotating accretion disk perturbed by FBM in density, and obtained the light curves and spectral slopes of IDV, their results shown that a magnetized disk subjected to FBM can produce some of the features observed in light curves.

It is the purpose of the present paper to consider a fractional Langevin equation driven by FBM to describe the stochastic oscillations of particles in accretion disks around black hole in relativistic framework, and obtain the basic physical parameters (displacement, velocity and luminosity) of the vertically oscillating disks by the means of the numerical calculation method. Then this model is used to explain some astrophysical observational features related to IDV.

2 The fractional Langevin equation with FBM for vertical oscillations of accretion disks

We assume that the source of IDV is placed within a geometrically thin, optically thick disk around a black hole, and the accretion disk in relativistic framework is perturbed by a stochastic process and a frictional force. Then the vertical oscillations of the particles in this disk can be described by a generalized relativistic Langevin equation, given by (Harko et al. 2012)

$$\frac{d^2z}{dt^2}+c\mu\int_0^t\gamma(t-\varsigma)\frac{dz(\varsigma)}{d\varsigma}d\varsigma+c^2\omega_\perp^2 z=\frac{c^2}{M_D}\xi^z(t)\,, \qquad (1)$$

where M_D and c denote the mass of the accretion disk and the speed of light in vacuum, z and

$\omega_\perp$ represent the displacement and frequency of particles' vertical oscillation, respectively. According to the literature (Harko et al. 2012), for a Kerr accretion disk, $\omega_\perp$ can be described as an expression of the gravitation radius ρ, the mass M and spin parameter a_* of center black hole. $\gamma(t)$ is the memory-damped kernel function which satisfies the fluctuation-dissipation theorem (Kursawe et al. 2013)

$$\left\langle \xi^z(t)\xi^z(\varsigma)\right\rangle = \frac{1}{\beta}\gamma(t-\varsigma), \tag{2}$$

where β is constant, $\langle\cdot\rangle$ represents the ensemble average. $\xi^z(t)$ is the fractional Gaussian noise, which is the general derivative of FBM $\xi^z(t)=\frac{dB_H(t)}{dt}$, has zero mean $\left\langle \xi^z(t)\right\rangle = 0$. $B_H(t)$ is a normalized Gaussian process (Mandelbrot & Van Ness 1968), and defined as

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{0}[(t-s)^{H-1/2}-(-s)^{H-1/2}]dB(s)\right) + \frac{1}{\Gamma(H+1/2)}\left(\int_0^t (t-s)^{H-1/2}dB(s)\right), \tag{3}$$

where the so-called Hurst exponent H is a constant in (0,1), and Γ represents the special function $\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx$.

For H in the range 0<H<1, FBM has the property of autocorrelation

$$\left\langle B_H(t)B_H(\varsigma)\right\rangle = \frac{1}{2}\left(t^{2H}+\varsigma^{2H}-|t-\varsigma|^{2H}\right) \quad . \tag{4}$$

So the fractional Gaussian noise can be written as：

$$\xi^z(t) = \frac{dB_H(t)}{dt} = \frac{1}{\Gamma(H-1/2)}\int_{-\infty}^{t}|t-s|^{H-3/2}dB(s) \quad . \tag{5}$$

Then Eq.(2) can be expressed as

$$\left\langle \xi^z(t)\xi^z(\varsigma)\right\rangle = \frac{1}{\beta}\gamma(t-\varsigma) = \frac{1}{\beta}2H(2H-1)|t-\varsigma|^{2H-2} \quad , \tag{6}$$

exhibiting a power-law decay with scaling exponent 2H-2 of the difference between the two times t and ς. It demonstrates that fractional Gaussian noise is a stationary process. When H<1/2, the noise correlator has a negative sign, the process is antipersistence or anti-correlated. While positive correlations or persistence occur for H>1/2, and the motion becomes superdiffusive. Normal Brownian noise without correlation corresponds to the limit H=1/2, and H=1 is the limiting case of ballistic (fully persistent) motion.

In Eq.(6), the power-law tail of the fractional Gaussian noise is matched by the power-law kernel inside the memory integral expression. For this power-law form, Eq.(1) becomes a fractional Langevin equation

$$\frac{d^2 z}{dt^2} + c\mu\Gamma(2H+1)\frac{d^{2-2H}}{dt^{2-2H}}z(t) + c^2\omega_\perp^2 z = \frac{c^2}{M_D}\xi^z(t) \tag{7}$$

where

$$\frac{d^{2-2H}}{dt^{2-2H}}z(t) = \frac{1}{\Gamma(2H-1)}\int_0^t (t-\varsigma)^{-(2-2H)}\frac{dz(\varsigma)}{d\varsigma}d\varsigma\,. \tag{8}$$

It is the fractional derivative of displacement in Caputo sense (Burov & Barkai 2008). The definition of fractional derivative is introduced in Appendix. In Eq(7), the memory integral diverges for H smaller than 1/2, such that the H is restricted to the range $1/2<H<1$.

The fractional derivative of displacement represents the weighted integral for velocity, and the variation of weight is associated with the power-law index 2-2H decided by the FBM. When the H tends to 1/2, the damped kernel function and fractional Gaussian noise degenerate into $\delta(t)$ function (Gaussian white noise), and their memories disappear. However, when H tends to 1, the damped kernel function becomes constant, the memories of system velocity at each time point are completely the same, and the fractional Gaussian noise has long-rang dependence property.

By introducing a set of dimensionless parameters defined as $z=(c^2/M_D)Z$, $a=c\mu\Gamma(2H+1)$, $\omega=c\omega_\perp$, we may rewrite the Eq.(1) as

$$\frac{d^2 Z}{dt^2} + a\frac{d^{2-2H}}{dt^{2-2H}}Z + \omega^2 Z = \xi^z(t)\,, \tag{9}$$

The total energy of particles is the sum of kinetic plus potential energy, can be written as

$$E(t) = \frac{1}{2}\left(\frac{dz(t)}{dt}\right)^2 + \frac{1}{2}c^2\omega_\perp^2 z(t)^2\,. \tag{10}$$

And the output luminosity L represents the energy loss of the disk due to viscous dissipation and the presence of stochastic force, may be written in dimensionless form as

$$L(t) = -\frac{dE(t)}{dt} = \frac{c^4}{M_d^2}\bar{L}(t)\,, \tag{11}$$

where

$$\bar{L}(t) = -\frac{dZ(t)}{dt}\cdot\left[\frac{d^2 Z(t)}{dt^2} + \omega^2 Z(t)\right] = -\frac{dZ(t)}{dt}\cdot\left[-a\frac{d^{2-2H}Z(t)}{dt^{2-2H}} + \xi^z(t)\right]. \tag{12}$$

The steady-state mean autocorrelation function of dimensionless luminosity $\bar{L}$ can be described as

$$R_L(\tau) = \frac{\langle (\bar{L}(t)-\mu_L)(\bar{L}(t+\tau)-\mu_L) \rangle}{\sigma_L^2} \quad , \tag{13}$$

where μ_L and σ_L^2 represent the mean and variance of $\bar{L}(t)$, respectively. By Fourier transform, the Eq.(13) may be translated into the PSD of oscillating luminosity as

$$P(f) = \int_{-\infty}^{+\infty} R_L(\tau) e^{-i2\pi f \tau} d\tau \, . \tag{14}$$

3 the numerical approach to the fractional Langevin equation

In this section, we introduce a numerical procedure to solve the dimensionless the fractional Langevin equation, and obtain the displacements, velocities and luminosities of accretion disk oscillations.

At first, by the method of order reduction, the Eq.(9) can be convert to a set of multivariable fractional-order differential equations as follow:

$$\frac{d^{2-2H} Z}{dt^{2-2H}} = U \quad , \tag{15}$$

$$\frac{d^{2H-1} U}{dt^{2H-1}} = V \, , \tag{16}$$

$$\frac{dV}{dt} = -aU - \omega^2 Z + \xi^z(t) \, . \tag{17}$$

Then we assume that the time t is discretized with timestep *h*. When 1/2<*H*<1, Caputo fractional derivative of *U* and *Z* may be written as differential approximations forms according to Eq.(A8). So the above stochastic differential equations (15)-(17) can be changed to Eulerian difference schemes. And the values of these variables at each timestep *m* are approximate calculated as

$$Z_m = h^{2-2H} U_{m-1} - \sum_{k=1}^{m} \sigma_k^{(2-2H)} Z_{m-k} \ , \tag{18}$$

$$U_m = h^{2H-1} V_{m-1} - \sum_{k=1}^{m} \sigma_k^{(2H-1)} U_{m-k} \ , \tag{19}$$

$$V_m = V_{m-1} + (-aU_m - \omega^2 Z_m + \eta(m))h \quad , \tag{20}$$

where the expression of $\sigma_k^{(.)}$ is shown as Eq.(A9), and $\eta(m)$ is a fractional Gaussian noise series generated by filtering white noise method developed by Li & Chi (2003),

$$\eta(m) = w(m) * h(m) \, , \tag{21}$$

where $*$ means the operation of convolution, and $w(m)$ presents Gaussian white-noise function. Its expression is

$$w(m) = IFFT\{\exp[j \cdot \pi(1 - rnd(0,1))]^{-2\pi}\}, \tag{22}$$

where $rnd(0,1)$ is uniformly distributed number within (0,1). FFT presents the operator of Fourier transform, IFFT is the inverse of FFT.

In Eq.(21), *h(m)* is an impulse function of a linear filter, and is expressed as follow:

$$h(m) = IFFT\{[FFT(r_m)]^{0.5}\}, \tag{23}$$

where r_m presents the autocorrelation of fractional Gaussian noise, and can be written as

$$r_m = [(m+1)^{2H} - 2m^{2H} + (m-1)^{2H}]/2 \quad . \tag{24}$$

Solving the Eq.(18)-(24) by iterative computation, we can obtain the values of the displacement Z, speed V and U. Then the dimensionless form of the disk luminosity can be written as

$$\bar{L}(m) = -V_m \cdot [-aU_m + \eta(m)] \tag{25}$$

4 Stochastic oscillations of the general relativistic disks

By means of the above numerical approach, we may obtian the basic physical parameters (displacement, velocity, and luminosity) of the vertically oscillating accretion disks in Kerr geometries, on which the influence of the exponent *H* will be discussed.

For an accretion disk around a supermassive black hole with the mass of M=$10^8 M_{sun}$, we fixed the gravitation radius *n*=7, the spin parameter of center black hole $a_* = 0.9$, and μ=0.03, β=1/6, respectively, and adopt the value of integration step *h*=0.1, initial displacement *Z*(0)=0, initial velocity *v*(0)=1, Then we varied the parameter *H* and solve the Eqs. (18) and (25). Then the behaviors of *Z*, *V* and $\bar{L}$ of the vertically oscillating disks can be obtained by numerical simulations, and are presented in Figs1,2 and 3.

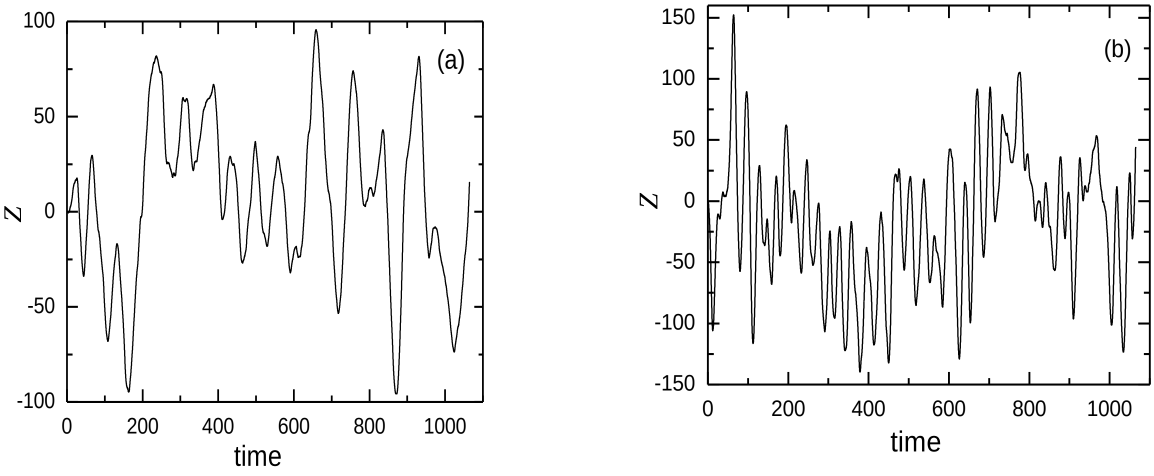

Fig.1 Vertical displacement of the stochastically oscillating disk for different values of H:

(a) *H*=0.6, (b)*H*=0.9.

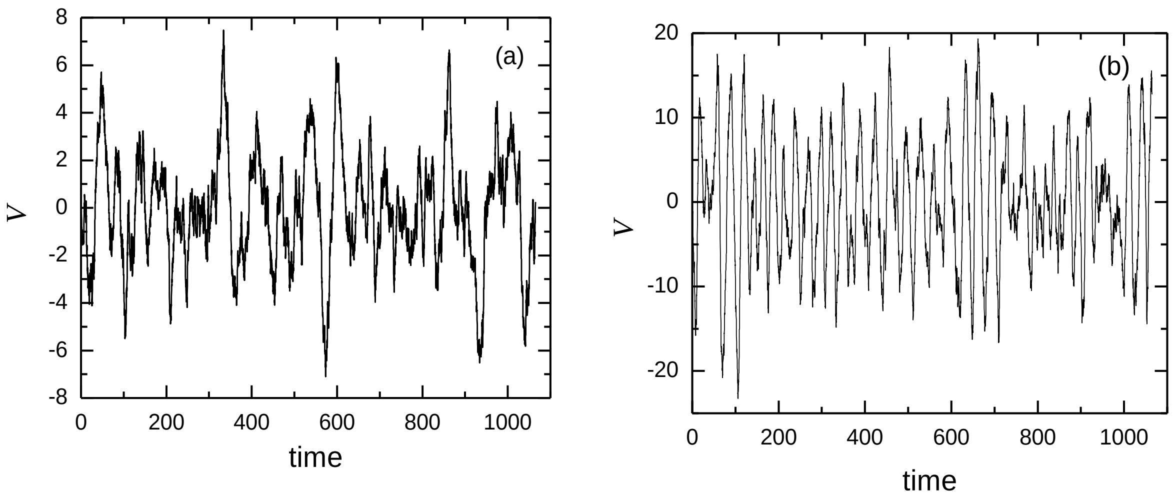

Fig.2 Velocity of the stochastically oscillating disk for different values of H:

(a)H=0.6, (b)H=0.9.

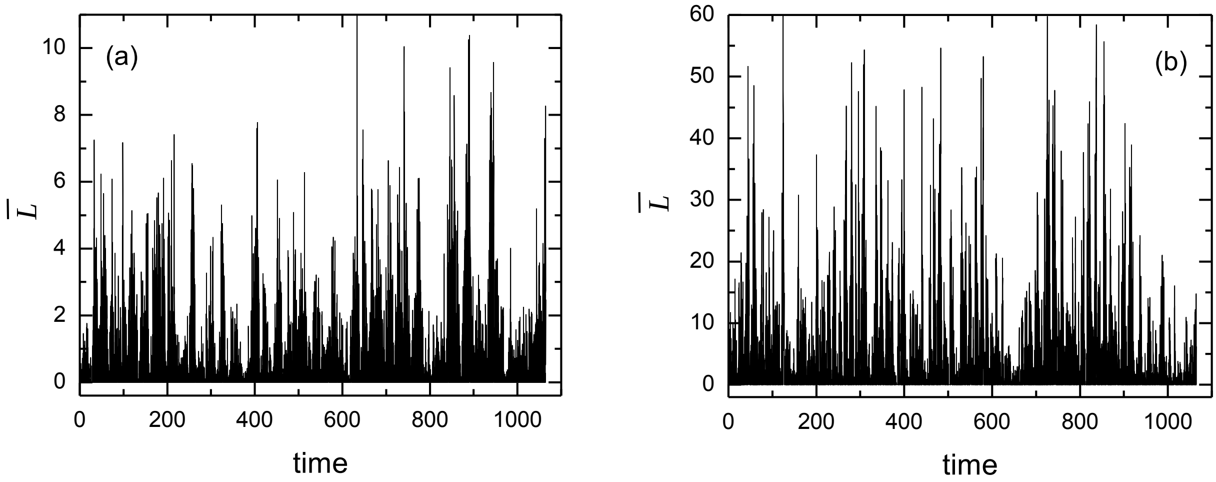

Fig.3 Luminosity of the stochastically oscillating disk for different values of H:

(a) *H*=0.6, (b) *H*=0.9.

Those Figures show that the variations of *Z*, *V* and $\overline{L}$ with time are different when *H* exponent of FBM takes different value. Compare the case of *H*=0.9 with the case of H=0.6, the frequencies of quasi-periodic changes of the displacements and velocities are higher, and their amplitudes are greater. So the total energy of disk oscillation is higher too, and more energy loss makes the amplitudes of output luminosity $\overline{L}$ greater.

In order to describe the stochastic characteristics of oscillating disks, we plot the PSDs of the luminosities and the sample fit lines in the cases of H=0.6 and H=0.9, as shown in Fig.4. The black curve represents the points produced by taking the PSDs of the simulated light curve, the red lines are the fits according to the polynomial assumption, and the blue lines represent the power-law fits of PSDs at high frequencies, respectively. The power-law fit is a power plus a constant given by Vanghan (2010), $P(f) = \varepsilon f^{-\alpha} + \lambda$ with three parameters, where α denotes power-law index, the power-law normalization ε and the additive constant λ are constrained to be non-negativ.

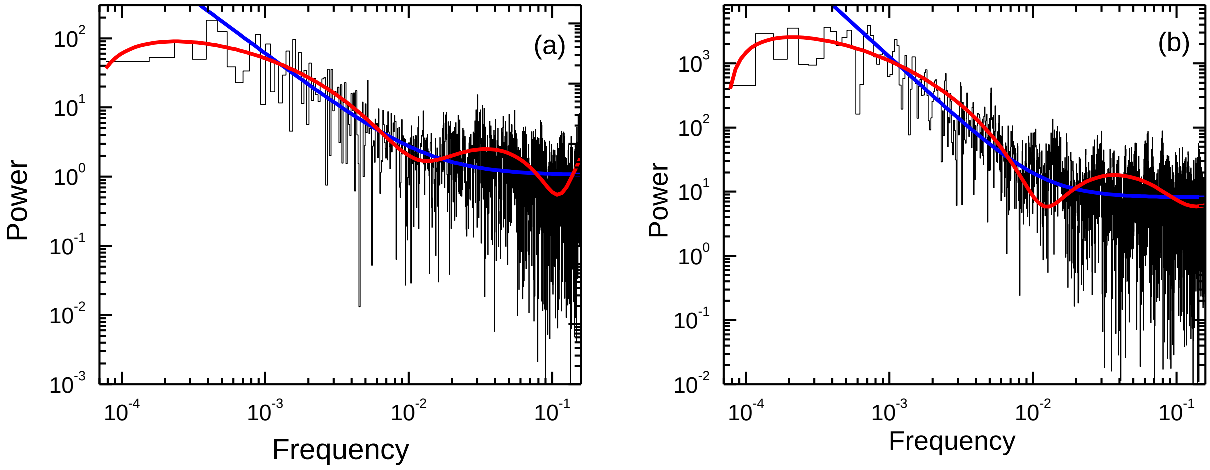

Fig.4 PSD of luminosity of the stochastically oscillating disk for different values of H: (a)*H*=0.6, (b) *H*=0.9.

As we known that the PSD of Galactic black hole binaries and AGNs is generally modeled with a power-law, and the power-law index (namely spectral slope) α keeps a constant value in a certain range of spectral frequency *f*. At high frequencies, the PSDs present a steep slope. For the IDV of the BLLac Objects, observations have manifested that most of the spectral slope α is in the interval 1.5 to 2. When the accretion disks are subjected to a fixed damping and white Gaussian noise, Harko and Mocanu (2012) have demonstrated the range for the spectral slope of their oscillating luminosity is from 1.35 to 1.65 because of the different gravitation radius and spin parameter. It is roughly the same with the result computed by our present model in the case of *H* tending to 1/2 as shown in Fig.4(a), in which the steep slope of the PSD is $\alpha = 1.5433 \pm 0.00278$ in the case of *H*=0.6. And we can find that as the *H* exponent increases, the order (2-2H) of the fractional displacement derivative is decreased, but the value of α is increased, and the PSD curve is more steep. When *H*=0.9, $\alpha = 2.0397 \pm 0.00119$. This result is well consistent with the statistical properties of the observed IDV light curves mentioned above.

5 Summary and Discussion

In the present paper, we have considered a fractional Langevin equation with fractional Gaussian noise to describe the vertical oscillations of general relativistic disks around a black hole, and obtained the displacements, velocities and luminosities of oscillating disks by means of numerical calculation method, then analyzed the properties of the PSDs for different *H* exponent. The main results show that as the *H* exponent is increased, the memories of damped kernel function and fractional Gaussian noise can be enhanced, then make the energy and luminosity of oscillating disk increased, and the spectral slope at high frequencies in PSD curves is increased. In the restricted range of *H*. the range of spectral slope α is well consistent with the observations of IDV presenting in the optical domain of output luminosity. So as a possible astrophysical application, this toy model may turn out to be very useful in explaining IDV phenomenon.

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Appendix: Review of fractional derivative

In this appendix, we will present the several definitions of fractional derivatives, and obtain the differential approximation for Caputo derivative. We begin by introducing the concept of the fractional calculus.

Let $f(t):[a,b]\to R$ be a time-function. Then the fractional calculus can be denoted by a fundamental operator ${}_aD_b^{\nu}$ where:

$$ {}_aD_b^{\nu} f(t) = \frac{d^{\nu} f(t)}{dt^{\nu}} = f^{(\nu)}(t) \quad \text{(A1)} $$

Where ν is the order of the derivative, and it is a real number satisfying $n-1 \le \nu \le n$ ($n \in N$). If $\nu > 0$, the operator ${}_aD_b^{\nu}$ expresses fractional derivative, and it is fractional integral when $\nu < 0$.

Based on the concept of fractional calculus, there are a lot of definitions for the fractional derivative. Here we only introduce the Grünwald-Letnikov definition, Riemann-Liouville definition, and Caputo definition.

1. The Grünwald-Letnikov fractional derivative of order ν of function *f(t)* at time-instant *t* is defined by (MacDonald et al. 2015)

$$ {}_0^GD_t^{\nu} f(t) = \lim_{h\to 0} h^{-\nu} \sum_{m=0}^{x/h} (-1)^m \binom{\nu}{m} f(x-mh), \quad \text{(A2)} $$

where the expression of the binomial coefficient $\binom{\nu}{m}$ gives

$$ \binom{\nu}{m} = \frac{\Gamma(\nu+1)}{m!(\nu+1-m)} \quad \text{(A3)} $$

We can find that the computation of a Grünwald-Letnikov derivative requires knowledge of the entire past history of the system, this means that the fractional derivative is no local but global.

2. The Riemann-Liouville fractional derivatives including the left and right derivatives of the order $\nu > 0$, are given by (Almeida & Torres 2011)

$$ {}_0^R D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_0^t \frac{f(x)}{(t-x)^{\nu+1-n}} dx , \tag{A4} $$

and

$$ {}_t^R D_0^{\nu} f(t) = \frac{(-1)^n}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_t^0 \frac{f(x)}{(x-t)^{\nu+1-n}} dx , \tag{A5} $$

respectively. It can be found that in the case of $\nu \in N$, the Riemann-Liouville fractional derivatives coincide with the standard integral-order derivative of *f(t)*.

3. the left and right Caputo fractional derivatives of order ν are defined by (Almeida & Torres 2011)

$$ {}_0^C D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^n(x)}{(t-x)^{\nu+1-n}} dx \tag{A6} $$

and

$$ {}_t^C D_0^{\nu} f(t) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_t^0 \frac{f^n(x)}{(x-t)^{\nu+1-n}} dx \tag{A7} $$

respectively.

If $\nu = n$, or $n-1 < \nu < n$,and the function *f(t)* satisfies $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$, there exists a equivalence relation between the Grünwald-Letnikov, the Riemann-Liouville and the Caputo fractional derivatives, namely ${}_0^C D_t^{\nu} f(t) = {}_0^R D_t^{\nu} f(t) = {}_0^G D_t^{\nu} f(t)$. According to this relation, when $0 < \nu < 1$, the differential approximation for Caputo derivative of *f(t)* can be written as (Dimitrov 2015)

$$ {}_0^C D_t^{\nu} f(t_m) \approx \frac{h^{-\nu}}{\Gamma(n-\nu)} \sum_{k=0}^{m} \sigma_k^{(\nu)} f(x-kh) \tag{A8} $$

where *h* is timestep, $t_m = mh$, and

$$ \sigma_k^{(\nu)} = (-1)^k \binom{\nu}{k}, \ (k=0,1,2\ldots), \quad \binom{\nu}{k} = \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!} \tag{A9} $$

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