Communication Complexity

A treasure house of lower bounds

Prahladh Harsha
TIFR

Communication Complexity Model [Yao]

$$f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$$

Akbar

Birbal

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x

$\Downarrow$

Akbar

y

$\Downarrow$

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$$\longleftarrow$$

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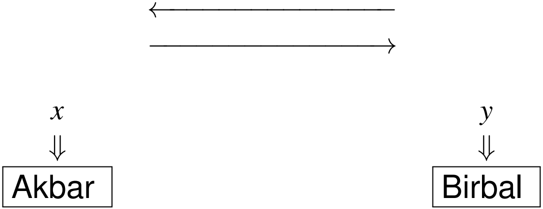

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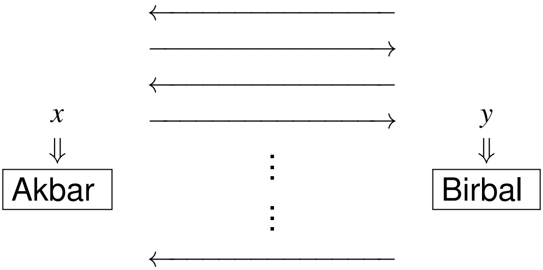

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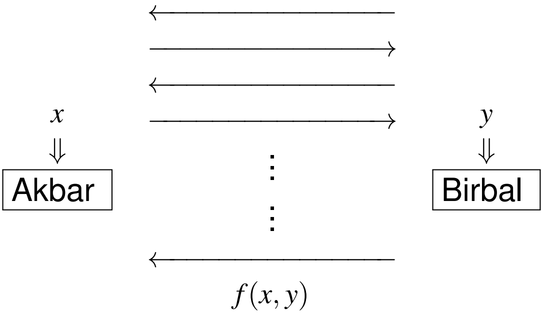

Question: How many bits must Akbar and Birbal exchange to compute f?

Example I: Equality

$$EQ(x_1x_2\ldots x_n, y_1y_2\ldots y_n) = \begin{cases} 1 & \forall i, x_i = y_i \\ 0 & \text{otherwise} \end{cases}$$

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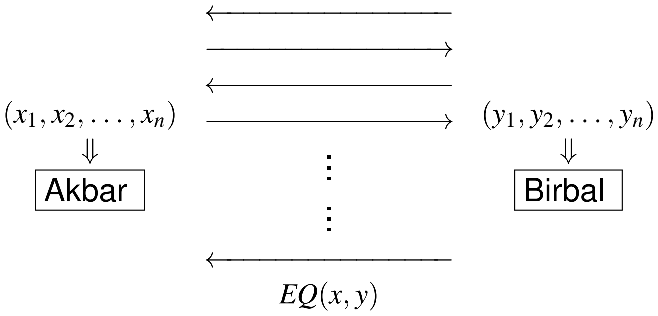

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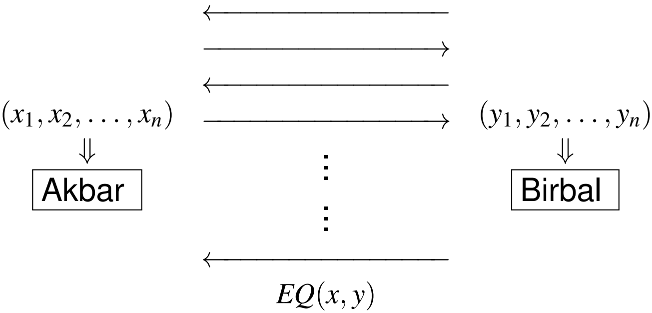

Deterministic Communication Complexity of EQ: n

Example I: Equality (Contd)

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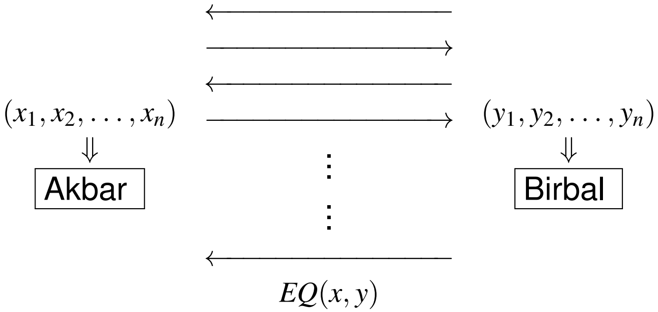

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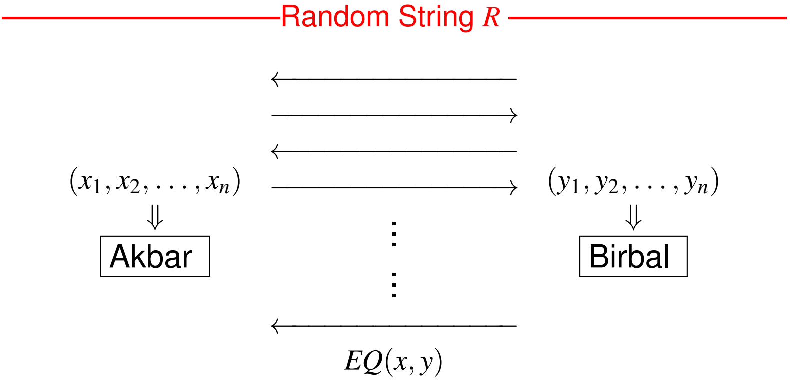

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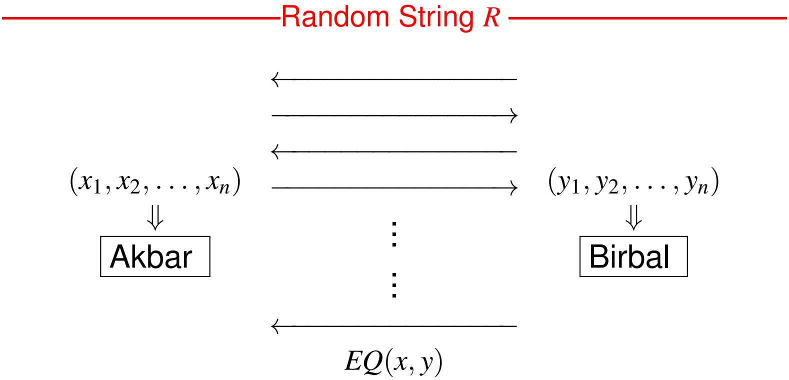

Randomized Communication Complexity of EQ: constant !

Example II: Set Intersection

$$INTER(x_1x_2\ldots x_n, y_1y_2\ldots y_n) = \begin{cases} 1 & \exists i, x_i = y_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

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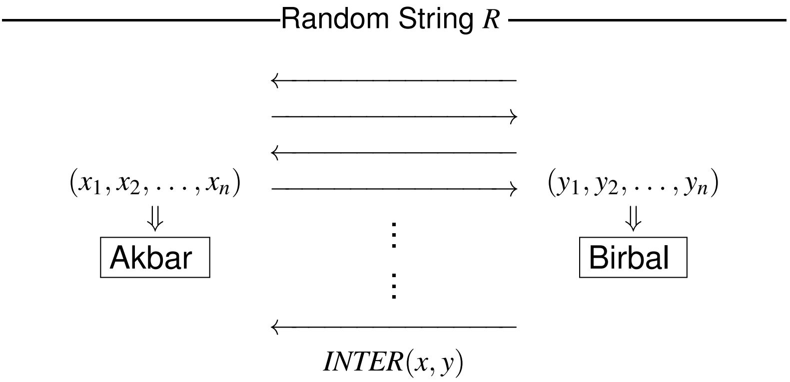

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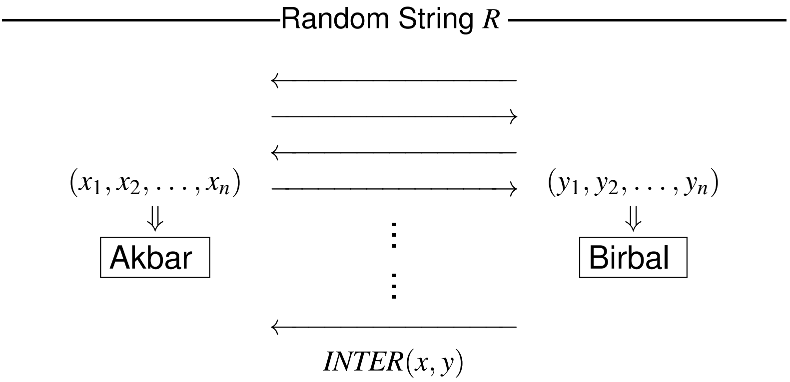

Randomized Communication Complexity of $INTER$: $\Omega(n)$

Example III: Tribes

$$TRIBES(\bar{x_1}\bar{x_2}\ldots\bar{x_n},\bar{y_1}\bar{y_2}\ldots\bar{y_n}) = \begin{cases} 1 & \forall i, INTER(\bar{x_i},\bar{y_i}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

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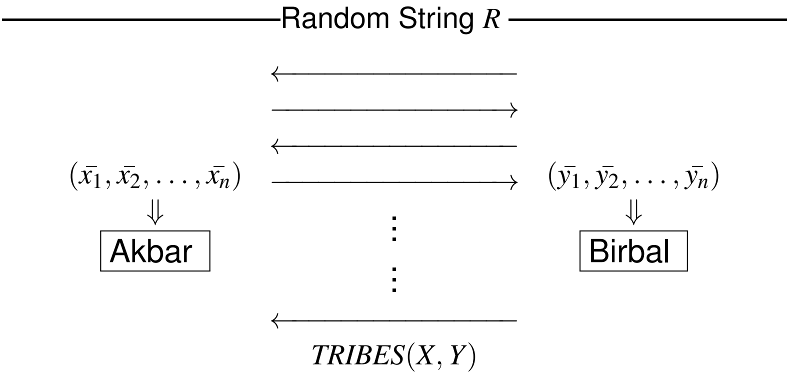

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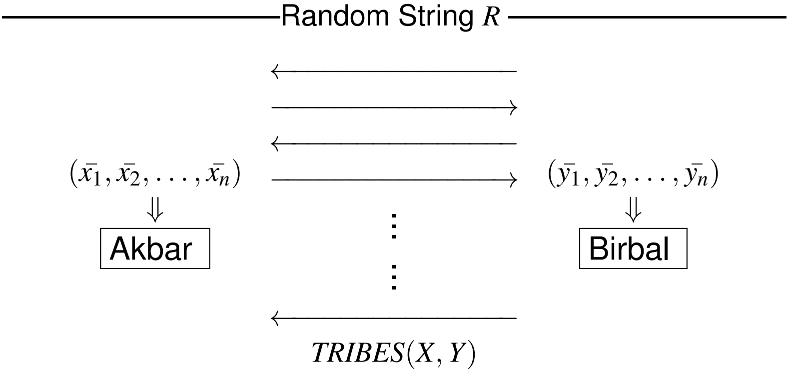

Randomized Communication Complexity of $TRIBES$: $\Omega(n)$

Applications

Data structures, VLSI design, time-space tradeoffs, circuit complexity, streaming, auctions, combinatorial optimization ...

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Data structures, VLSI design, time-space tradeoffs, circuit complexity, streaming, auctions, combinatorial optimization ...

Randomized Communication Complexity of *INTER*: $\Omega(n)$

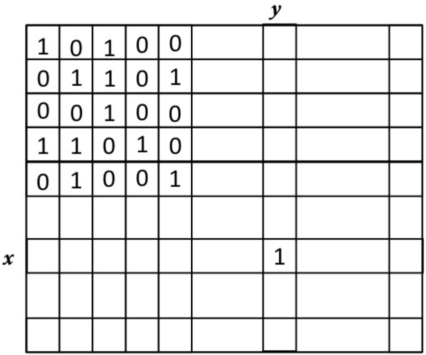
- There is no parallelizable monotone circuit that computes a matching in a given graph
- There is no polynomial sized extended formulation for the travelling salesperson problem

How does one prove such lower bounds?

Matrix of inputs

1	0	1	0	0		
0	1	1	0	1		
0	0	1	0	0		
1	1	0	1	0		
0	1	0	0	1		

Matrix of inputs



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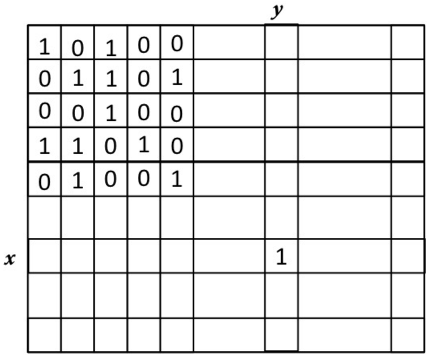
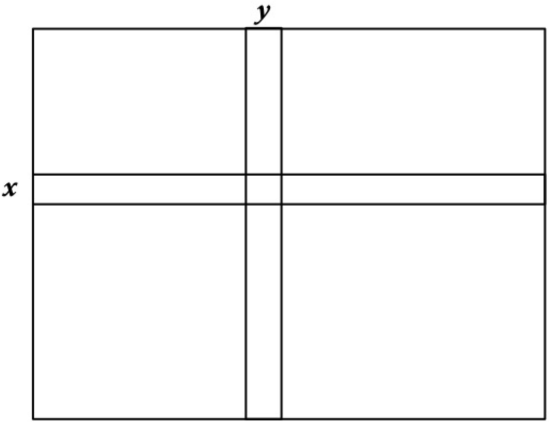


Property of matrix: rank, discrepancy can be used to prove lower bounds

It is all about rectangles . . .



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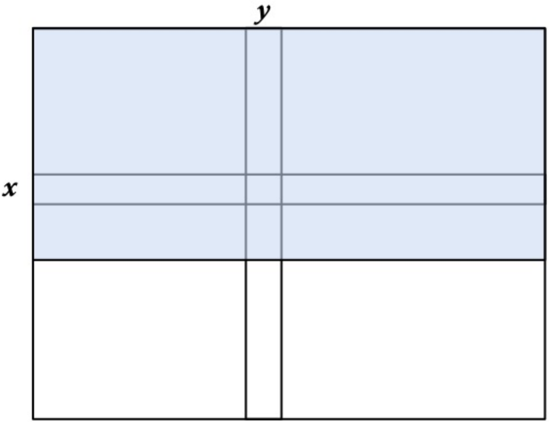

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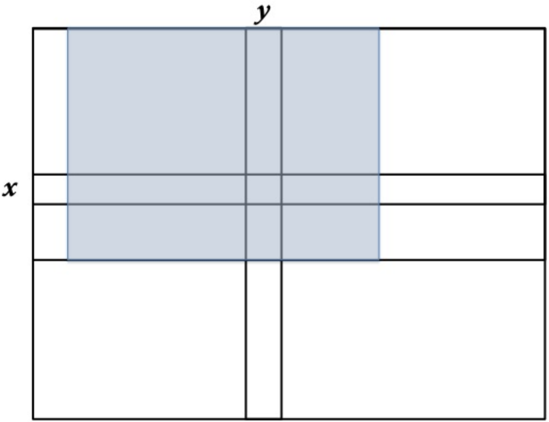

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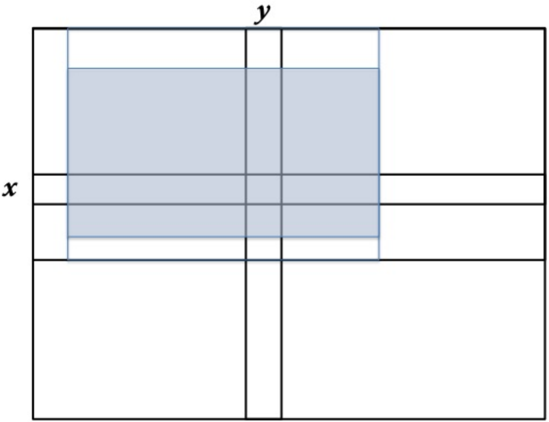

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- Lots of communications implies small rectangles
- Low communication implies large rectangles

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[Razborov] Any large rectangle for *INTER* has nearly equal number of 1's and 0's
Corollary: Comm. Complexity of *INTER* is large

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- Tribes has large monochromatic rectangles
- Rectangle method does not yield bound for *TRIBES*
- However, *TRIBES* is “close” to another function which has large rectange bound
- Above observations suffices to show *TRIBES* has large comm. complexity

Thank You