Invariant Theory (IT) & Standard Monomial Theory (SMT)

K. N. Raghavan, IMSc, Chennai

06 July 2013

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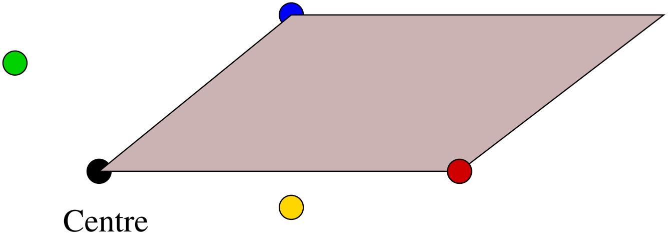

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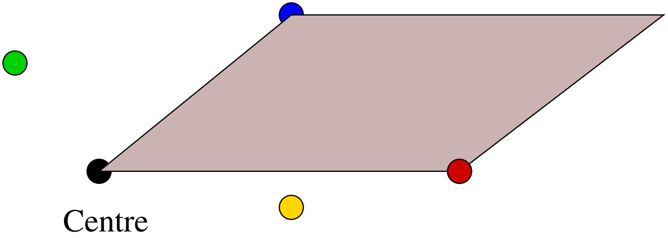

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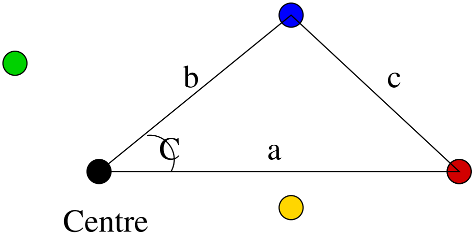

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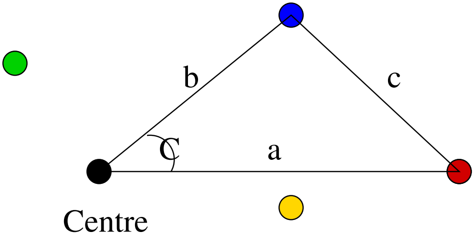

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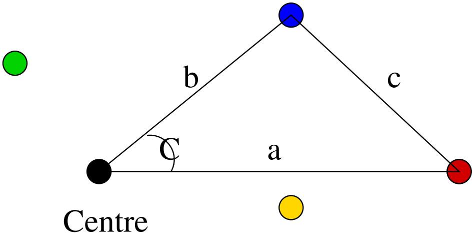

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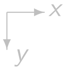

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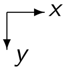

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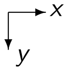

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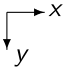

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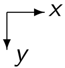

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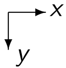

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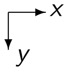

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