Event–Related Synchronization/Desynchronization in Coupled Nonlinear Oscillator Systems and Applications

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Aim

- To present a general coupled nonlinear oscillator model for event-related synchronization/desynchronization (ESR/ERD)
- To explain relevant experiment in brain dynamics
Plan of talk

1. Introduction
2. Experimental realization of ERD/ERS
3. A general model for ERD/ERS
4. ERD/ERS in an array of coupled Stuart-Landau oscillators
5. ERD/ERS in other interesting coupled oscillators
6. Discussion & Outlook
Basic structure of neuronal systems & brain waves

- Human brain is one of the most complex systems
- Made of billions of neurons/glial cells
- Information processing & transmission
Basic structure of neuronal systems & brain waves

- Dynamics of a single neuron
- Membrane potential - oscillates when external disturbance occurs
Basic structure of neuronal systems & brain waves

Nerve impulse propagation: Propagation of electrical pulses along axons

- Hodgin-Huxley equation
- FitzHugh-Nagumo equation
- Bonhoeffer van der Pol equation
- Hindmarsh Rose equation
Basic structure of neuronal systems & brain waves

- Brain waves - EEG data - Cognition/Motion/Function

### Different stages of activity & EEG brain-waves - Classical view

<table>
<thead>
<tr>
<th>Stage</th>
<th>Frequency (Hz)</th>
<th>Wave type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher mental activity, including perception and</td>
<td>~ 35–50</td>
<td>γ</td>
</tr>
<tr>
<td>consciousness.</td>
<td></td>
<td></td>
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<tr>
<td>High stressful situations, difficult mental focus</td>
<td>13–30</td>
<td>β</td>
</tr>
<tr>
<td>and concentrations.</td>
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<tr>
<td>Wakefulness with relaxed and effortless alertness.</td>
<td>8–12</td>
<td>α</td>
</tr>
<tr>
<td>Light sleep and drowsiness.</td>
<td>4–8</td>
<td>θ</td>
</tr>
<tr>
<td>Deepest stages of dreamless sleep.</td>
<td>&lt;4</td>
<td>δ</td>
</tr>
</tbody>
</table>

- Synchronization ⇒ act of concurrence or the adjustment of rhythms of two mutually interacting systems; Ex. a pair of coupled oscillators
Basic structure of neuronal systems & brain waves

Event–related synchronization/desynchronization In the brain

- Event–related oscillatory responses of the brain - characterized by ERD/ERS of neuronal oscillations.
- ERD $\Rightarrow$ relative decrease in the intensity of a specific frequency band
- ERS $\Rightarrow$ increase in the intensity of a specific frequency band.
- ERS/ERD $\Rightarrow$ important in deciding the accomplishment of a behavioral/functional task.
Experiment an anesthetized rats

- 10 male wistar rats anaesthetized with ketamine xylazine.
- Negative pinch test $\Rightarrow$ deep anaesthesia (first 45 mins)
- Positive pinch test $\Rightarrow$ light anaesthesia (last 25 mins)

First 45 mins. $\delta$ (0-4 Hz) ERS; Last 25 mins. $\delta$ ERD & $\theta$ (4-8 Hz) ERS

(Left) ERS/ERD in $\delta$ band during deep/light anaesthesia. (Right) ERS in $\theta$ band during light anaesthesia.

A general coupled nonlinear oscillator model for ERD/ERS

First macroscopic model for the ERD phenomenon

\[ \dot{X}_j = F(X_j, \epsilon_j) + \frac{A}{N} \sum_{k=1}^{N} G_{ik}(X_k - X_j) + B Y, \quad (\cdot = \frac{d}{dt}) \]
\[ \dot{Y} = G(Y, \epsilon_e), j = 1, 2, \ldots N. \]

- \( F(X_j, \epsilon_j) \) ⇒ the nonlinear limit cycle behavior of the \( j \)th uncoupled oscillator (neuron) and \( \epsilon_j \) ⇒ corresponding system parameter.
- \( A \) ⇒ coupling strength of the neurons.
- \( Y \) ⇒ state vector of the external stimulus signal and \( \epsilon_e \) ⇒ control parameter of the external oscillator

Results

- External stimulus $B$ typically affects synchronization in the system of coupled oscillators.

- Suppose $X_j = X$, $j = 1, 2, \ldots, N$ due to $A$ (when $B = 0$): Sufficient strength of $B$ causes desynchronization $\Rightarrow$ ERD.

- When the system is partially synchronized, depending upon the strength of $B$ the desynchronized fraction of oscillators (either all or some of them) entrain with the synchronized group $\Rightarrow$ synchrony is enhanced $\Rightarrow$ ERS.

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ERD/ERS in an array of coupled Stuart-Landau oscillators

- \( \dot{z}_j = (a + i\omega_j - (1 + ic)|z_j|^2)z_j + \frac{A}{N} \sum_{k=1}^{N} (z_k - z_j) + Bz_e, \)
- \( \dot{z}_e = (a_e + i\omega_e - (1 + ic_e)|z_e|^2)z_e. \)

- \( c \) - nonisochronicity parameter, \( a \) - Hopf bifurcation parameter,
- \( z_j = x_j + iy_j \) - complex amplitude of the \( j \)th oscillator
- \( \omega_j \) - natural frequency of the oscillators distributed according to a Lorentzian distribution

Lorentzian distribution: \( g(\omega) = \frac{\gamma}{\pi} \left[ \gamma^2 + (\omega - \omega_0)^2 \right]^{-1} \)

\( \gamma \) \( \Rightarrow \) half width at half maximum, \( \omega_0 \) \( \Rightarrow \) central frequency.
ERD/ERS in an array of coupled Stuart-Landau oscillators

Occurrence of ERD/ERS

Left - time evolution of $x$ of all the oscillators. Right - corresponding phase portraits on $(x_i, y_i)$ plane. $A = 1.1$ and (top) $B = 0.01$, (mid) $B = 0.08$, (bottom) $B = 0.35$.

- Mid panel - ERD - Major synchronized group and few desynchronized oscillators.

Model results and the experiment on rats

Numerical illustration of ERD and ERS in a system of 1000 SL oscillators for different stimulus strengths:

Comparison with the experiment on rats

- (a) ⇒ the state of deep anaesthesia - synchronization in δ band.
- (b) Around 45 mins. B becomes strong enough to cause ERD δ and ERS in θ band.
Measure of ERD/ERS

Intensity of Synchronization

\[ I = \langle |e^{i\theta_j}| \rangle = \frac{1}{T} \int_0^T \left| \frac{1}{N} \sum_{j=1}^N e^{i\theta_j} \right| dt, \]

\[ \theta_j = \tan^{-1}(y_j/x_j) \Rightarrow \text{phase of the } j\text{th oscillator} \]

- \( I = 0 \) \( \Rightarrow \) no synchronization (or) incoherence (\( \theta_j \neq \theta_i \), \( i, j = 1, 2, ... N; i \neq j \))
- \( I = 1 \) \( \Rightarrow \) complete synchronization (\( \theta_j = \theta_i \), \( i, j = 1, 2, ... N; i \neq j \))
- \( 0 < I < 1 \) \( \Rightarrow \) partial synchronization
- The more oscillators that are in synchrony the higher will be the Intensity.
Measure of ERD/ERS

Change in the intensity $I$

There is a critical strength of external stimulus for ERD

ERD occurs for a finite window of $B$ - depending on $A$. 
Stability of the ERD/ERS state

- Major synchronized group - $N_1$
- Separated group - few oscillators $N_2$ compared to $N_1$.
- Size ratio of the two cluster state - $r : 1 - r(= N_1/N : N_2/N)$
- $r >> 0 \sim 1$ for the desynchronized state.

Dynamics of large group $u$ and small group $v$

\[
\begin{align*}
\dot{u} & \approx ((a + i\omega) - (1 + ic)|u|^2)u + \text{Be}^{i\omega t}, \\
\dot{v} & \approx ((a + A + i\omega) - (1 + ic)|v|^2)v + Au + \text{Be}^{i\omega t}.
\end{align*}
\]
**Analytical Support**

- Assume: major synchronized group is completely synchronized with the external force $z_e = e^{i\omega_e t}$, while the small group is not.

- $\Rightarrow \omega = \omega_e + c$ and $a = 1 - B$.

- The dynamics of the small group ($v$) -
  \[
  \dot{w} = ((a + \hat{A} + ic) - (1 + ic)|w|^2)w + \hat{A} + B,
  \]

- $w = ve^{-i\omega_e t}$.

- $w_1 = 1$ - stable fixed point - complete synchronization - $u = v$.

- Equation for $|w|^2$: $\Rightarrow (|w|^2 - 1)(p|w|^4 + q|w|^2 + s) = 0$

  \[
  p = (1 + c^2), \quad q = (2(A + B) - (1 + c^2)), \quad s = (A + B)^2
  \]

- Other two fixed points $w_2$ and $w_3$ determine the stability of the desynchronized state.
Analytical Support

Fixed points $w_2$ and $w_3$:

$$|w_{2,3}|^2 = \frac{1 + c^2 - 2(A + B) \pm \sqrt{(1 + c^2 - 2(A + B))^2 - 4(1 + c^2)(A + B)^2}}{2(1 + c^2)}$$

Fixed points $w_2$ and $w_3$ exist for

$$B < B_I = \frac{(1 + c^2)(\sqrt{1 + c^2} - 1)/(2c^2) - A,}{B_I}$$

$B_I$ is a saddle-node bifurcation point.

Determinant of the Jacobian matrix for $w_2$ and $w_3$

$$\det(J_{w_2,3}) = \frac{\mp 2(A + B)(A + B + 2)\triangle}{(1 + c^2 + 2(A + B))} \pm \triangle,$$

$$\triangle = \left((1 + c^2 + 2(A + B))^2 - 4(1 + c^2)(A + B)(A + B + 2)\right)^{\frac{1}{2}}$$
Analytical Support

- $\det(J_{w_{2,3}})$ is always negative (positive) for $w_2$ ($w_3$).
- $w_2 \Rightarrow$ saddle; $w_3 \Rightarrow$ either an unstable node or focus for $|c| \leq 1$.
- When $|c| > 1$ and $B > B_{II}$, $w_3$ is either a stable node or focus.

$B_{II}$ is a Hopf bifurcation point given by

$$B_{II} = \frac{(1 + c^2)}{(\sqrt{4 + (1 + c^2)^2 + 2})} - A,$$

which is determined from the condition $\text{tr}(J_{w_3}) = 0$.

$$\text{tr}(J_{w_3}) = 2 \frac{(1 - c^2)(A + B) + \triangle}{(1 + c^2)}.$$
Analytical Support

$B - |c|$ phase diagram. $B_I$ (solid line) and $B_{II}$ (dashed line) are saddle-node and Hopf bifurcation boundaries respectively. $B_{III}$ (dotted line) is obtained numerically by solving $w$ equation.

- ERD does not occur in region IV - only synchronized solutions are stable
- Desync. solutions exist in regions II and III
ERD/ERS in systems with other forms of coupling

Delay coupled SL:
\[ \dot{z}_j = (a + i \omega_j - (1 + i c)|z_j|^2)z_j + \frac{A}{N} \sum_{k=1}^{N} (z_k(t - \tau) - z_j) + Bz_e \]

Weighted coupled SL:
\[ \dot{z}_j = (a + i \omega_j - (1 + i c)|z_j|^2)z_j + \sum_{k=1}^{N} A_{jk} (z_k - z_j) + Bz_e \]

\( \tau \) - time delay parameter, \( A_{ij} \) - random coupling matrix.

ERD/ERS in (Left) Delay coupled (Right) Weighted coupled SL system.
Coupled Rössler oscillators

\[
\begin{align*}
\dot{x}_j &= -\omega_j y_j - z_j + Bx_e, \\
\dot{y}_j &= \omega_j x_j + ay_j + \frac{A}{N} \sum_{k=1}^{N} (y_k - y_j), \\
\dot{z}_j &= b + z_j(x_j - c).
\end{align*}
\]

The external stimulus described by

\[
\begin{align*}
\dot{x}_e &= -\omega_e y_e - z_e, \\
\dot{y}_e &= \omega_j x_e + ay_e, \\
\dot{z}_e &= b_e + z_e(x_e - c_e).
\end{align*}
\]

\(a, b \text{ and } c \Rightarrow \text{System parameters that determine the periodicity/chaoticity of the system.}\)
ERD/ERS in a system of coupled chaotic oscillators

Occurrence of ERD and ERS in a system of 1000 coupled Rössler oscillators.
ERS and ERD in bi-directionally coupled SL oscillators

\[
\begin{align*}
\dot{z}_j &= (1 + i \omega - (1 + ic)|z_j|^2)z_j + \frac{A}{N} \sum_{k=1}^{N} (z_k - z_j) + Bz_e, \\
\dot{z}_e &= (1 + i \omega_e - (1 + ic)|z_e|^2)z_e + \frac{B}{N} \sum_{k=1}^{N} z_k,
\end{align*}
\]

Occurrence of ERD in a system of 1000 SL oscillators
ERS and ERD in Phase Model

\[
\dot{\theta}_i = \omega_i + \frac{A}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i - \beta) + B \sin(\theta_e - \theta_i - \beta)
\]

\[
\dot{\theta}_e = \omega_e + \frac{B}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_e - \beta)
\]

ERD and ERS in a system of Phase Model \((x_i = \cos(\theta_i)\) and \(y_i = \sin(\theta_i)\))
Thermodynamic limit

- In the limit of infinite number of oscillators
- Order parameter: \( \Lambda = \frac{1}{N} \sum_{j=1}^{N} \exp[\theta_j - \theta_e] \)

Order parameter Equation (Ott and Antonsen ansatz):

\[
\dot{\Lambda} = -(\gamma - i\bar{\omega})\Lambda - \left( \frac{A}{2} |\Lambda|^2 \Lambda + \frac{B}{2} (\Lambda^2 - |\Lambda|^2) \right) e^{i\beta} \\
+ (\frac{A}{2} \Lambda + \frac{B}{2} (1 - \Lambda^2)) e^{-i\beta}.
\]

Occurrence of ERD and ERS \((K = B/A)\).
ERD in firing neuronal model

FitzHugh-Nagumo oscillator:

\[
\dot{u}_j = \mu(u_j - v_j - \frac{u_j^3}{3} + I(t)) + A \frac{1}{N} \sum_{k=1}^{N} (u_k - u_j) + Bu_e,
\]

\[
\dot{v}_j = u_j - b_j v_j + a_j + A \frac{1}{N} \sum_{k=1}^{N} (v_k - v_j) + Bv_e
\]

External oscillator:

\[
\dot{u}_e = \mu(u_e - v_e - \frac{u_e^3}{3} + I(t)),
\]

\[
\dot{v}_e = u_e - b_e v_e + a_e
\]

Occurrence of ERD in a system of 1000 FitzHugh-Nagumo oscillators

Discussion & Outlook

- Diffusively coupled oscillators in the presence of external field serve as models to explain ERD/ERS.
- ERD/ERS phenomenon generic to all kinds of diffusively coupled models.
- Form of the external stimulus signal - immaterial.
- Model can also account for (de)synchronization behaviour due to cognition and pathalogies.
- Model can be used to find ways to avoid desynchronization in spin torque nano-oscillators (STNO).
References

Thank You!