Differential structures in C*-algebras

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1. A structural analogy between C^*-algebras and Uniform Banach algebras
2. Harmonic analysis on locally compact groups and semigroups with weights
3. Differential Structures in C^* algebras

Let M be a C^∞-manifold, assumed compact for simplicity.

Algebras encoding the structure of M are the following.

(1) $C(M) \rightarrow$ pointset topology —— Commutative C^*-algebra

(2) $C^\infty(M) \rightarrow$ differential structure——dense Frechet subalgebra

(3) $L^\infty(M), L^p(M) \rightarrow$integration structure —abelian von Neumann algebra

(4) $\Omega^*(M)$ de Rham algebra $\rightarrow$ homological structure

(5) $Lip(M)$ Lipschitz algebra $\rightarrow$ metric structure

Geometric structure on $M \rightarrow$ algebraic structure associated with M

(Gelfand-Naimark) A C^*-algebra A — topological data (a noncommutative virtual compact space)

Differential structure on A specified by a dense *-subalgebra B - functional analytic characterization of $C^\infty(M)$? -

regularity properties expected from B.

(1) spectral invariance

(2) closure under holomorphic functional calculus

(3) closure under C^∞ -functional calculus

(4) K-theory isomorphism

(5) hermiticity

(6) suitable complete locally convex topology preferably nuclear

(7) automatic continuity, extendability and domain invariance of morphism

(8) ideal structure $I \to I \cap B$

(9) derivation like structure

(10) admitting C^*-enveloping algebra

Aspects of Theory

(1) General theory – differential seminorm approach and growth conditions on seminorms

(2) Methods: smooth crossed products and deformation

(3) concrete examples of non commutative smooth algebras - non commutative Torus, non commutative R^n- non commutative cylinders and spheres-

A = a locally convex *-algebra

$C^*(A)$ = enveloping C^*-algebra, $j : A \to C^*(A)$ natural map

spectral seminorm $\{x : p(x) < 1\} \subset A^{qr}$, spectral invariance of A via j,

spectral representation π $sp_A(x) = sp_{C^*(\pi)}(\pi(x))$

Theorem

(JOT 2003; Rendi.Cir.Math.Palermo 1998) The following are equivalent.
(1) A is spectrally invariant.
(2) A is C^*-spectral.
(3) A is spectral and hermitian.
(4)A is local and $rad(A) = srad(A)$.
(5)A is spectral and stable.
(6) A admits a spectral continuous bounded operator representation on a Hilbert space.
(7)Every algebraically irreducible representation of A on a vector space is similar to a continuous algebraically irreducible *-representation on a Hilbert space.
(8) Every algebraically irreducible representation of A on a vector space extends to a topologically irreducible *-representation of $C^*(A)$ on a Hilbert space.

Theorem

(JOT 2003). Let A be a Frechet *-algebra each element of which is bounded. Let A be spectrally invarient. Then $K_*(A) = K_*(C^*(A))$.

This leads to unbounded spectral representation and unbounded *-seminorms.

Philosophy of unbounded operator representations

(1) naturality of unbounded representations of *-algebra

(2) Examples from Quantum Theory and group representations

(3) Pathologies and choice of well behaved representations; e.g. self - adjoint, standard, weakly unbounded, well behaved,

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A = a *-algebra

unbounded C^*-seminorm p in A having domain $D(p)$, $kerp = N_p$ and defining left ideal $\mathcal{N}_p = \{x \in D(p) : Ax \subset D(p)\}$.

A_p=Hausdorff completion of $D(p)/N_p$.

For $\Pi_p \in Rep(A_p)$, define an unbounded operator representation $(\pi_p, D(\pi_p), H)$ of A as

$D(\pi_p) = span\{\Pi_p(x + N_p)\psi : x \in \mathcal{N}_p\, \psi \in H_{\Pi_p}\}$

H_{π_p} =closure of $D(\pi_p)$

$\pi_p(a)(\Pi_p(x + N_p)\psi = \Pi_p(ax + N_p)\psi$

An unbounded representation $(\pi, D(\pi), H)$ is well behaved if there exists an unbounded C^*-seminorm p in A such that $\pi = \pi_p$ with $H = H_{\pi_p} = H_{\Pi_p}$.

Theorem

(JOT 2001; JMSJ 2004)
(1) p is hereditary spectral iff p is spectral and stable.
*(2) A admits a spectral well behaved *-representation iff A is spectrally invariant.*

Well behaved representations include

(1) standard representations of polynomial algebras

(2) integrable representations of universal enveloping algebra of Lie algebra

(3) standard representations of Heisenberg commutation relations

(4) Moyal quantization map of the Moyal algebra.

C^∞-spectral representations and C^∞-spectral seminorms?

Given $\pi : A \to B(H)$, $x = x^* \in A$ and $f \in C^\infty(sp(\pi(x)))$, there exists $y = y^* \in A$ such that $\pi(y) = f(\pi(x))$ and $sp_A(x) = sp_{C^*(\pi)}(\pi(x))$.

(approach to smooth algebras initiated by Blackadar and Cuntz)

Two steps :

(a) smooth structures defined by a differential norm

(b) take appropriate limits over differential norms

(a) $(\mathcal{U}, \|\cdot\|_0) = C^*$-normed algebra C^*-algebra completion A.

differential norm on $\mathcal{U}$

$T : x \in \mathcal{U} \to (T_k(x) \in$ non negative sequences

$T_0(x) \leq \|x\|_0$

$T(x+y) \leq T(x) + T(y)$

$T(\lambda x) = |\lambda| T(x)$

$T(xy) \leq T(x)T(y)$ convolution

$T(x) = 0$ implies $x = 0$.

In the absence of l^1-summability, take

$l^1(\mathcal{U}, T) = \{x \in \mathcal{U} : T_{tot}(x) < \infty\}$,$T_{tot}(x) = \sum T_k(x)$ normed algebra

$\mathcal{U}_T$ = completion differential Banach *-algebra

$p_k(x) = \sum_{i=0}^{i=k} T_i(x)$, $C^k(\mathcal{U}, T) = \widetilde{(\mathcal{U}, p_k)}$ completion = a Banach *-algebra

$C^\infty(\mathcal{U}, T) = \underleftarrow{\lim}_{k \to \infty} C^k(\mathcal{U}, T) = \mathcal{U}_\tau$ = a differential Frechet algebra

Theorem

(JOT 2011)
(1) $\mathcal{U}_\tau$ *is a* C^**-spectral algebra.*
(2) $\mathcal{U}_\tau$ *is spectrally invarient in* A.
(3) $\mathcal{U}_\tau$ *is a hermitian* Q*-algebra.*
(4) $\mathcal{U}_\tau$ *is closed under holomorphic functional calculus and* C^∞ *-functional calculus of self adjoint elements.*

Analytic and entire analytic structure defined by T given by the following sub algebras obtained by taking inverse limits and direct limits as $n \to \infty$.

$$C^\omega(\mathcal{U}, T) = \cup_{t>0} \mathcal{U}_{T(t)} = \lim_{\substack{\rightarrow \\ n\to\infty}} \mathcal{U}_{T(1/n)}$$

$$C^{e\omega}(\mathcal{U}, T) = \lim_{\substack{\leftarrow \\ n\to\infty}} \mathcal{U}_{T(n)}$$

$$C^\omega(\mathcal{U}_\tau, T) = \lim_{\substack{\rightarrow \\ n\to\infty}} \mathcal{U}^\omega_\tau(1/n) \text{ – lmc } Q\text{-algebra}$$

$$C^{e\omega}(\mathcal{U}_\tau, T) = \lim_{\substack{\leftarrow \\ n\to\infty}} \mathcal{U}^\omega_\tau(n) \text{ – Frechet algebra}$$

Here $\mathcal{U}^\omega_\tau(k) = l^1(\mathcal{U}^\omega_\tau, T(k))[T(k)_{tot}]$.

analytic seminorm p : $\limsup_{s\to\infty}\{\log(x_1x_2x_3...x_s)/s\} \leq 0$ for $\|x_i\|_0 \leq 1$.

T analytic on $\mathcal{U}$ if for some $t > 0$, $T(t)_{tot}$ is analytic on $l^1(\mathcal{U}, T(t))$; T is entire analytic if this holds for all $t > 0$.

$l^\omega = \inf\{t \text{ as above}\}$. $T(t)_k(x) = t^k T_k(x)$

$\widetilde{\mathcal{U}^\omega} = \cup_{t>l^\omega} \mathcal{U}_{T(t)}$ complete m-convex algebra.

$\widetilde{\mathcal{U}^{e\omega}} = \cap_{t>0} \mathcal{U}_{T(t)}$ a Frechet algebra.

Theorem

(1) If T is analytic on $\mathcal{U}$, then $\widetilde{\mathcal{U}^\omega}$ is C^*-spectral hermitian Q-algebra closed under holomorphic functional calculus of A.
(2) It T is entire analytic, similar conclusion holds for $\widetilde{\mathcal{U}^{e\omega}}$.

The analytic structure on $\mathcal{U}_\tau$ defined by T is described by the topological algebras $\mathcal{U}^\omega_\tau$ and $\mathcal{U}^{e\omega}_\tau$, and similar results hold for them.

(b) T is of total order $\leq k$ if for each T - bounded sequence $\{x_s\}$ in $\mathcal{U}$,

$\limsup_{s\to\infty} \log T_{tot}(x_1x_2x_3....x_s)/\log s \leq k$. i.e. $T_{tot}(x^s)\ O(s^k)$ for $s \to \infty$.

a derived norm α on $\mathcal{U}$ is the quotient norm of the total norm of a differential norm of total order $\leq k$ for some k.

Λ_{cd}= all closable derived norms on $\mathcal{U}$; $\Lambda_{cd}^{\leq k}$= closable derived norms of order $\leq k$.

smooth envelope of $\mathcal{U} = \mathcal{S}(\mathcal{U})$= completion of $(\mathcal{U}, \Lambda_{cd})$.

C^k*-envelope* of $\mathcal{U} = \mathcal{S}^k(\mathcal{U})$= completion of $(\mathcal{U}, \Lambda_{cd}^{\leq k})$.

Following chain of topological algebras

$\mathcal{U} \subset \mathcal{S}\mathcal{U} = \lim_{\leftarrow} \mathcal{S}^k\mathcal{U} \subset \mathcal{S}^{k+1}\mathcal{U} \subset \mathcal{S}^k\mathcal{U} \subset A$.

$\mathcal{U}$ is smooth if $\mathcal{U} = \mathcal{S}\mathcal{U}$.

C^k-completion $= C^k\mathcal{U}=$ completion of $\mathcal{U}$ in all closable flat differential norms of order $\leq k$.

C^∞-completion $= \cap_k C^k(\mathcal{U})$.

$\mathcal{U}$ is a C^∞-algebra if $\mathcal{U} = C^\infty\mathcal{U}$.

Chain of topological algebras

$\mathcal{U} \subset \mathcal{S}\mathcal{U} \subset C^\infty\mathcal{U} \subset C^{k+1}\mathcal{U} \subset C^k\mathcal{U} \subset A$.

Theorem

(JOT 2011) The smooth algebras and the C^∞ -algebras have the desired regularity properties.

Examples

(1) function algebras of smooth functions

$C^\infty[0,1], C^k[0,1], CBV[0,1], AC_p[0,1], W^{m,p}[0,1], Lip[0,1]$

$C_0^\infty(R), C_0^k(R), \mathcal{S}(R)$

(2)operator algebras defined by derivations

(a) Given a finite set of closed unbounded derivations in a C^*-algebra A,

$C^n(A)$=C^n-elements of A defined by these derivations,

$C^\infty(A)$-C^∞-elements

(b) Lie group G acting on a C^*-algebra A,

$C^\infty(A, \alpha), C^k(A, \alpha)$

(c) non commutative torus T^n_θ

(d) smooth operator algebra crossed product defined by an action of R / any Lie group?

(3) differential structures defined by almost commuting self adjoint operators as well as by an n-tuple of strongly commuting self adjoint operators

Programme : smooth compact operators – $\mathcal{S}(\mathbb{Z}^2)$ acting on $l^2(\mathbb{Z})$ - smooth trace class, smooth Hilbert-Schmidt and smooth von Neuman-Schatten class operators - Search for smooth bounded operators? - differential algebras of bounded operators?

(G, A, α) = a C^*-dynamical system

crossed product C^*-algebra $C^*(G, A, \alpha) = C^*(L^1(G, A)$,twisted convolution) encodes the C^*-dynamics

non commutative analogue of covariance algebra for G acting on a locally compact space.

smooth crossed product = a non commutative analogue of algebras of smooth functions encoding differential dynamics given by action of a Lie group G on a manifold M.

(Schwaitzer) a general method of constructing spectrally invariant sub algebras of crossed product C^*-algebras

Is there a non commutative smooth structure lurking behind?

Theorem

(PMSIASc 2006) Let α be a strongly continuos action of $\mathbb{R}$ by continuos *-automorphisms of a Frechet *-algebra A.
(a) Let A admits a bai contained in A^∞ (C^∞-elements)which is bai for the Frechet algebra A^∞. Then
$E(S(\mathbb{R}, A^\infty, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$
$= E(L^1(\mathbb{R}, A, \alpha))$ *if α is isometric.*
enveloping $\sigma - C^*$-algebra of smooth Schwartz Frechet algebra crossed product = continuous crossed product of enveloping C^*-algebra
(b) Let A be hermitian and Q.Then
$RK_*(S(\mathbb{R}, A^\infty, \alpha)) = K_*(C^*(\mathbb{R}, A, \alpha))$

α = an action of $\mathbb{R}$ on a C^*-algebra A leaving a dense *-sub algebra $\mathcal{U}$ invariant.

$\widetilde{\mathcal{U}}$ = α-invariant smooth envelope

=completion of $\mathcal{U}$ in α-invariant differential seminorms

(smooth Frechet analogue of Connes analogue of Thom isomorphism)

Theorem

(PMSIASc 2006)
(a) $RK_*(S(\mathbb{R}, \mathcal{U}_\tau^\infty, \alpha)) = K_{*+1}(A)$
(b) If $\widetilde{\mathcal{U}}$ *is metrizable, then*
$RK_*(S(\mathbb{R}, \mathcal{U}_\tau^\infty, \alpha)) = K_{*+1}(A)$.

Let A be a C^*-algebra.The Frechet algebras $S(\mathbb{R}, A, \alpha)$ and $S(\mathbb{R}, A^\infty, \alpha)$ are differential Frechet algebras; and are smooth sub algebras of the crossed product C^*-algebra $C^*(\mathbb{R}, A, \alpha)$.

Definition

Let $(A, \|\cdot\|_0)$ be a C^*-algebra. Let B be a dense *-subalgebra of A. Then B is called a *Frechet* (D^*_∞)*-subalgebra of* A if there exists a sequence of seminorms $\{\|\cdot\|_i : 0 \leq i < \infty\}$ such that the following hold.

1. For all $i, 1 \leq i < \infty$, for all x, y in B, $\|xy\|_i \leq \|x\|_i\|y\|_i, \|x^*\|_i = \|x\|_i$.
2. For each i, $1 \leq i < \infty$, there exists $D_i > 0$ such that $\|xy\|_i \leq D_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i)$ holds for all x, y in B.
3. B is a Hausdorff Frechet *-algebra with the topology τ defined by the seminorms $\{\|\cdot\|_i : 0 \leq i < \infty\}$.

(Kissin and Shulman) A Banach (D^*_k)-algebra is defined by a (D^*_k) family $\{\|\cdot\|_i : 0 \le i \le k\}$ with $(B, \|\cdot\|_k)$ a Banach *-algebra- n/c analogue of C^k-functions- Frechet (D^*_∞)-algebra proposed as n/c analogue of $C^\infty[a,b]$

Theorem

*Let $(B, \{\|\cdot\|_i\}_0^\infty)$ be a Frechet D^*_∞-subalgebra of a C^*-algebra $(A, \|\cdot\|_0)$. Then there exists a sequence $(B_k, \|\cdot\|_k)$ of dense Banach *-subalgebras of A such that the following hold.*

1. *Each B_k is a Banach (D^*_k)-subalgebra of A continuously embedded in A.*
2. *The sequence B_k forms an inverse limit sequence of Banach *-algebras and $B = \lim_{\leftarrow k\to\infty} B_k$, the inverse limit of B_k.*

A Frechet (D^*_∞) sub algebra of a C^*-algebra has properties analogous to $C^\infty[a, b]$.

(a) It can not be a Banach algebra under any norm.

(b) If a Banach algebra contains B, it must contain some B_k.

(c) The norm closed ideals of B are precisely the intersections with B of norm closed ideals of the C^*-algebra A.

(d) Its morphisms are continuous in the C^*-norm.

Theorem

A Frechet (D^*_∞)-sub algebra of a C^*-algebra has desired regularity properties.

A Frechet (D^*_1)-algebra is one in which the defining seminorms $\|\cdot\|_i$ satisfy the first order growth condition

$\|xy\|_i \leq \|x\|_0\|y\|_i + \|x\|_i\|y\|_0$.These smooth sub algebras of a C^*-algebra presumably have a richer structure.Examples include

(1) $C^\infty[a,b]$, $\{f \in C_0(R) : f^{'} \in C(R)\}$

(2) C^∞ elements of a C^*-algebra defined by a derivation that is a generator

(3) C^∞-domain of a closed unbounded multiplier on a C^*-algebra

(4) Certain algebras defined by Schatten-von Neumann classes as well as by Fredhom modules

A = a locally convex *-algebra/ a Frechet *-algebra.

Representation theoretic universal object for A can be constructed by two ways.

(1) In the frame work of representations into bounded Hilbert space operators, one gets a family of C^*-seminorms (Gelfand-Namark C^*-seminorms) corresponding to a defining family of seminorms. The Hausdorff completion produces a pro-C^*-algebra $E(A)$ universal for continuous bounded operator representations.

(2) In the frame work of unbounded operator representation theory, one takes direct sum π_u of unbounded GNS representations defined by states and produce a universal unbounded operator algebra $O(A)$.

Theorem

(PMSIASc 2001; JMAA 2007)
(1) Let A be a Frechet *-algebra. Then $E(A)$ is the completion of $O(A)$. Then A is an algebra with a C^*-enveloping algebra iff Every operator representation of A map A necessarily into bounded operators.
(2) Let A be a complete locally m-convex *-algebra. The following are equivalent.
(i) A admits a greatest continuous C^*-semi norm.
(ii) The hermitian spectral radius is dominated by a continuous semi norm.

$(A, \|\cdot\|) = C^*$-normed algebra + Banach *-algebra with norm $|.|$.
$\widetilde{A} = C^*$-algebra completion
$A \bigotimes A$ an A-bimodule
$d : A \to A \bigotimes A, da := 1 \bigotimes a - a \bigotimes a$ derivation
$\Omega^1 A :=$ sub module generated by
$\{adb = a \bigotimes b - ab \bigotimes 1 : a, b \in A\}$
$\Omega^k(A) := \Omega^1 A \bigotimes_A \Omega^1 A \bigotimes_A ...... \bigotimes_A \Omega^1 A$ k-times
$\Omega^* A := \bigoplus_{n=0}^{\infty} \Omega^n A$ abstract non commutative differential forms over A
graded *-algebra with derivation
$d(a_0 da_1 da_2 ... da_n) = da_0 da_1 da_2 .... da_n$

For $r \in R^+$, $|\omega = \sum \omega_k|_r := \sum r^k |\omega_k|_\pi$ norms on $\Omega^* A$
$\Omega^r A =$ completion of $(\Omega^* A, |.|_r)$ Banach *-algebra
(Arveson) $\Omega_\infty A := \varprojlim_{r \to \infty} \Omega_r A$ inverse limit
(Connes) $\Omega_\epsilon A := \varinjlim_{r \to 0} \Omega_r A$ direct limit

Theorem

(PMSIASc 2008)
(1) The bounded part of the Frechet algebra $\Omega_\infty A$ coincides with the Banach *-algebra A; and there exists a continuous *-homomorphism from $E(\Omega_\infty A)$ to $\widetilde{A}$.
(2) The algebra $\Omega_\epsilon A$ is a spectral m-convex Q-algebra; and $E(\Omega_\epsilon A) = C^*(A)$.

Given a K-cycle $(\pi, H.D)$ with $\pi : A \to B(H)$ a representation of A, π extends as a representation of $\pi : \Omega^* A \to B(H)$,let $J_0 = \ker \pi$ and $J = J_0 + dJ_0$, $\Omega^*_D = \Omega^* A/J$; viz.

$\Omega^k_D = \pi(\Omega^k A)/\pi(d(J_0 \cup \Omega^{k-1} A))$.

$\Omega^*_D = \bigoplus_{k=0}^{k=\infty} \Omega^k_D$ ... n/c de Rham algebra

Assume A to be closed under holomorphic functional calculus of $\widetilde{A}$.The algebra Ω^*_D can be topologized in several ways.

(a) $\|\cdot\|_{k,\pi}$= projective tensor product norm on Ω^k_A.
$\|\cdot\|_{\pi,q}$ = quotient norm on Ω^k_D.
$\Omega_{r,\pi}(A,D)$ = the completion of Ω^*_D in $\|\omega\| = \sum r^k \|\omega\|_{\pi,q}$
$\Omega^h_{r,\pi}(A,D)$ = functional calculus closure of Ω^*_D.
Then taking limits

$$\Omega^h_{\infty,\pi}(A,D) := \lim_{\leftarrow\, r\to\infty} \Omega^h_{r,\pi}(A,D)$$
$$\subset \lim_{\leftarrow\, r\to\infty} \Omega_{r,\pi}(A,D) = \Omega_{\infty,\pi}(A,D)$$
$$\Omega^h_{\epsilon,\pi}(A,D) := \lim_{\rightarrow\, r\to 0} \Omega^h_{r,\pi}(A,D)$$
$$\subset \lim_{\rightarrow\, r\to 0} \Omega_{r,\pi}(A,D) = \Omega_{\epsilon,\pi}(A,D)$$

(b) $\|\cdot\|_q$= quotient norm on Ω^k_D from the operator norm

$\Omega_r(A,D)$ = Banach *-algebra obtained by completing Ω^*_D in the corresponding norm

Taking holomorphic functional calculus closure and appropriate limits, we get

$$\Omega^h_\infty(A,D) := \lim_{\leftarrow \, r\to\infty} \Omega^h_r(A,D) \subset \lim_{\leftarrow \, r\to\infty} \Omega_r(A,D) = \Omega_\infty(A,D)$$

$$\Omega^h_\epsilon(A,D) := \lim_{\rightarrow \, r\to 0} \Omega^h_r(A,D) \subset \lim_{\rightarrow \, r\to 0} \Omega_r(A,D) = \Omega_\epsilon(A,D)$$

Theorem

(PMSIASc, 2008)
(1) $\Omega^h_\epsilon(A, D)$ *is Q-algebra spactrally invariant in* $\Omega_\epsilon(A, D)$ *and having* $\widetilde{A}$ *as its enveloping* C^**-algebra.*
(2) $\Omega^h_\infty(A, D)$ *(respectively* $\Omega^h_{\infty,\pi}(A, D)$*) is closed under the holomorphic functional calculus of* $\Omega_\infty(A, D)$ *(respectively* $\Omega_{\infty,\pi}(A, D)$*).*

Quantized Integrals(TAMS, 1999; PMSIASc, 2008)

Given a spectral triple (π, H, D) on a *-algebra A, various quantized integrals on $\Omega^* A$ like d-dimensional volume integrals and infinite dimensional integrals are defined using Dixmier trace depending on growth conditions on spectral triple. A unified approach to these integrals can be developed using quasi weights and the integrals might be extended to limit algebras like $\Omega_\infty A$ and $\Omega_\epsilon A$.

a positive linear functional on A - a non commutative analogue of complex Borel measure necessarily finite.

a weight on a von Meumann algebra - non commutative analogue of infinite positive measure

a quasi weight on A is tailored to suit unbounded operator algebras

a subspace N of A,

For a sub space N of A, let

$P(N) = \{\sum_{finite} x_k^* x_k; x_k \in A\}$

a weight $\phi : P(A) \to R_+ \cup \{\infty\}$ satisfying additivity and positive

Let $\mathcal{N}$ be a left ideal of A.

A quasi weight on $P(\mathcal{N})$is a map $\phi : P(\mathcal{N}) \to R_+$ that is additive and positive homogenious.Then $\mathcal{N} = \mathcal{N}_\phi$.

Given a quasi weight $(\phi, \mathcal{N}_\phi)$ on a *-algebra A, GNS construction can be carried on with it resulting into a strongly cyclic unbounded operator representation $(\pi_\phi, D(\pi_\phi), H_\phi)$ of A.

ϕ is admissible if π_ϕ represents A into bounded operators.

Non C^*-like phenomema of weak admissibility and strict inadmissibility.

(1) Quasi weight on a smooth sub algebra of a C^*-algebra is admissible.

(2) A quasi weight on an unbounded operator algebra defined by a weighted trace is strictly inadmissuble. In perticular, this holds for equilibrium states for BCS-Bogolubov model and interacting Bosons.

(3) The quasi weight defined on $\Omega^* A$ by the Dixmier trace is admissible.

(4) The finite dimensional volume integral on A extends as an admissible quasi weight on $\Omega_\infty A$; and the GNS representation so defined is unitarily equivalent to extension of the left action of A.

S closed symmetric operator with a dense domain $D(S)$in a Hilbert space $\mathcal{H}$.

$\mathcal{B}(\mathcal{H}) = C^*$-algebra of bounded operators

$\mathcal{K}(\mathcal{H}) = C^*$-algebra of compact operators on $\mathcal{H}$.

first order differential structure

(Kissin and Shulman)

$\mathcal{A}^1_S$ ={$A \in \mathcal{B}(\mathcal{H}) : AD(S) \subset D(S), A^*D(S) \subset D(S),$
$(SA - AS)^- \in \mathcal{B}(\mathcal{H})$.

$A_S := (SA - AS)^-$.

$\mathcal{A}^1_S$ = a Banach *-algebras with norm $\|A\|_1 := \|A\| + \|A_S\|$, $\| \cdot \|$ denoting the operator norm.

$\mathcal{U}_S$ be the C^*-algebra obtained by completing $\mathcal{A}^1_S$ in $\| \cdot \|$.

$\mathcal{U}_S$ — analogue of $C[a, b]$

$\mathcal{A}^1_S$—analogue of $Lip[a, b]$

$\mathcal{K}^1_S := \mathcal{A}^1_S \cap \mathcal{K}(\mathcal{H})$;

first order differential structure

(Kissin and Shulman)

$\mathcal{A}^1_S$ $=\{A \in \mathcal{B}(\mathcal{H}) : AD(S) \subset D(S), A^*D(S) \subset D(S),$ $(SA - AS)^- \in \mathcal{B}(\mathcal{H})$.

$A_S := (SA - AS)^-$.

$\mathcal{A}^1_S$ = a Banach *-algebras with norm $\|A\|_1 := \|A\| + \|A_S\|$, $\|\cdot\|$ denoting the operator norm.

$\mathcal{U}_S$ be the C^*-algebra obtained by completing $\mathcal{A}^1_S$ in $\|\cdot\|$.

$\mathcal{U}_S$ — analogue of $C[a, b]$

$\mathcal{A}^1_S$—analogue of $Lip[a, b]$

$\mathcal{K}^1_S := \mathcal{A}^1_S \cap \mathcal{K}(\mathcal{H})$;

$\mathcal{J}_S^1 := \{A \in \mathcal{K}(\mathcal{H}) : A_S \in \mathcal{K}(\mathcal{H})\}$,

$\mathcal{F}_S^1$ be the closure in the norm $\|\cdot\|_1$ of all finite rank operators in $\mathcal{A}_S^1$.

$\mathcal{A}_S^1$ is a *Banach D-algebra*; a dense *-sub algebra of a C^*-algebra satisfying $\|TR\|_1 \leq D(\|T\|_1\|R\| + \|T\|\|R\|_1)$ for all T, R in $\mathcal{A}_S$.

$\mathcal{F}_S^1 \subset \mathcal{J}_S^1 \subset \mathcal{K}_S^1 \subset \mathcal{A}_S^1$.

These algebras represent the non commutative Lipschitz structure of order 1 defined by S.

The non commutative C^1- structure defined by S as follows.

$\mathcal{A}_S^{(1)} := \{A \in \mathcal{U}_S : AD(S) \subset D(S), A^*D(S) \subset D(S), (SA - AS)^- \in \mathcal{U}_S\}$,

$\mathcal{K}_S^{(1)} := \mathcal{K}(\mathcal{H}) \cap \mathcal{A}_S^{(1)}$,

$\mathcal{J}_S^{(1)} := \{A \in \mathcal{K}_S^{(1)} : A_S \in \mathcal{K}_S^{(1)}\}$, and

$\mathcal{F}_S^{(1)} = \| \cdot \|_1$-closure of finite rank operators in $\mathcal{A}_S^{(1)}$.

These exhibit the first order differential structure defined by S described in terms of the derivation δ_S formally defined by S as $\delta_S(A) := i(SA - AS)^-$and considered in the C^*-algebra $\mathcal{U}_S$ as well as the in von Neumann algebra $\mathcal{M}_S$ generated by S.

second order differential structure

$\mathcal{A}_S^2 := \{A \in \mathcal{A}_S^1 : \delta_S(A) \in \mathcal{A}_S^1\}$, a Banach *-algebra with norm $\|A\|_2 = \|A\| + \|\delta_S(A)\| + (1/2!)\|\delta_S^2(A)\|$;

$\mathcal{K}_S^2 = \mathcal{A}_S^2 \cap \mathcal{K}(\mathcal{H})$;

$\mathcal{J}_S^2 = \{A \in \mathcal{K}_S^1 : \delta_S^1 \in \mathcal{J}_S^1\}$,

$\mathcal{F}_S^2 =$ closure in $\|\cdot\|_2$ of finite rank operators in $\mathcal{A}_S^2$.

Notice that for A in $\mathcal{A}_S^2$, $\delta_S(A) \in \mathcal{U}_S$; and thus the algebra $\mathcal{A}_S^2$ corresponds to the algebra of C^1-functions whose derivative is Lipschitzian.

The analogues of the algebra of C^2-functions are given as follows.
$\mathcal{A}_S^{(2)} = \{A \in \mathcal{A}_S^{(1)} : \delta_S(A) \in \mathcal{A}_S^{(1)}\}$ a closed sub algebra of $\mathcal{A}_S^2$,
$\mathcal{K}_S^{(2)} = \mathcal{A}_S^{(2)} \cap \mathcal{K}(\mathcal{H})$,
$\mathcal{J}_S^{(2)} = \{A \in \mathcal{K}_S^{(2)} : \delta_S(A) \in \mathcal{K}_S^{(2)}\}$,
$\mathcal{F}_S^{(2)} =$ closure in $\mathcal{A}_S^{(2)}$ of finite rank operators in $\mathcal{A}_S^{(2)}$.

Thus the non commutative second order differential structure defined by A is manifested as the following complex of Banach algebras which are dense smooth sub algebras of C^*-algebras.

$$\begin{array}{ccccccccc}
\mathcal{A}_S^{(2)} & \subset & \mathcal{A}_S^2 & \subset & \mathcal{A}_S^{(1)} & \subset & \mathcal{A}_S^1 & \subset & \mathcal{U}_S \\
\bigcup & & \bigcup & & \bigcup & & \bigcup & & \\
\mathcal{K}_S^{(2)} & \subset & \mathcal{K}_S^2 & \subset & \mathcal{K}_S^{(1)} & \subset & \mathcal{K}_S^1 & & \\
\bigcup & & \bigcup & & \bigcup & & \bigcup & & \\
\mathcal{J}_S^{(2)} & \subset & \mathcal{J}_S^2 & \subset & \mathcal{J}_S^{(1)} & \subset & \mathcal{J}_S^1 & & \\
\bigcup & & \bigcup & & \bigcup & & \bigcup & & \\
\mathcal{F}_S^{(2)} & \subset & \mathcal{F}_S^2 & \subset & \mathcal{F}_S^{(1)} & \subset & \mathcal{F}_S^1 & &
\end{array}$$

Proposition

*The Banach *-algebras* $(\mathcal{A}_S^2), \|\cdot\|_2)$,$(\mathcal{K}_S^2, \|\cdot\|_2)$,$(\mathcal{J}_S^2, \|\cdot\|_2)$ *and* $(\mathcal{F}_S^2, \|\cdot\|_2)$ *are semisimple;* $\mathcal{F}_S^2$ *has no closed two sided ideals; and* $\mathcal{F}_S^2 \subset \mathcal{I}$ *for any closed* **-ideal* $\mathcal{I}$ *of* $(\mathcal{A}_S^2, \|\cdot\|_2)$.

We consider the Lipschtz structure defined by $\mathcal{A}_S^1$ and $\mathcal{A}_S^2$.

$\mathcal{M} \subset \mathcal{N}$ be von Neumann algebras with same unit.

A W^**-derivation* $\delta : \mathcal{M} \to \mathcal{N}$ = an unbounded linear map whose domain dom (δ) is a unital *-sub algebra of $\mathcal{M}$

(i) dom (δ) is ultra weakly dense in $\mathcal{M}$

(ii) the graph of δ is ultra weakly closed in $\mathcal{M} \bigoplus \mathcal{N}$ and

(iii) δ is a *-derivation.

$dom(\delta)$ = a W^*-domain algebra.

(Weaver) a W^*-domain algebra = non commutative metric space

It is a Banach *-algebra with norm $\|x\|_1 := \|x\| + \|\delta(x)\|$.

Let $\mathcal{M}_S := W^*(\mathcal{U}_S)$ the von Neumann algebra generated by the C^*-algebra $\mathcal{U}_S$.

Notice that $\mathcal{M}_S = W^*(\mathcal{A}_S^1), \mathcal{U}_S = C^*(\mathcal{A}_S^1)$.

Proposition

Let S be as above.
(1) The derivation $\delta_S : \mathcal{M}_S \to \mathcal{B}(\mathcal{H})$ with domain $dom(\delta_S) = \mathcal{A}_S^1$ is a W^*-derivation.
(2) The Banach *-algebra $\mathcal{A}_S^1$ is dual of a Banach space; and the weak*-topology σ^1 on $\mathcal{A}_S^1$ is described as $A_\alpha \to A$ in weak*-if and only if $A_\alpha \to A$ ultra weakly in $\mathcal{M}$ and $\delta_S(A_\alpha) \to \delta_S(A)$ ultra weakly in $\mathcal{B}(\mathcal{H})$.

Theorem

Let S be as above.
(1) Let $X = X^* \in \mathcal{A}^1_S$. let $f \in$Lip$(sp(X))$.Let $\delta_S(X)$ commutes with X. Then $f(X) \in \mathcal{A}^1_S$ and $\|\delta_S(f(X))\| \leq L(f)\|\delta_S(X)\|$.
(2) Let $\mathcal{J}$ be a σ^1-closed *-ideal of $\mathcal{A}^1_S$. Then $\mathcal{J}$ is the σ^1-closure of $(\mathcal{J})^2$, where $(\mathcal{J})^2$ is the linear span of $\{AB : A \in \mathcal{J}, B \in \mathcal{J}\}$.
(3) Let $\mathcal{J}$ be a *-ideal of $\mathcal{A}^1_S$. Then $\delta_S(\mathcal{J})$ is contained in the ultra weak closure of $\mathcal{J}$ in $\mathcal{B}(\mathcal{H})$.
(4) Let $\mathcal{I}$ and $\mathcal{J}$ be *-ideals of $\mathcal{A}^1_S$. Then $\mathcal{I} \cap \mathcal{J}$ is contained in the σ^1 -closure of $\mathcal{I}\mathcal{J}$; and if $\mathcal{I}$ and $\mathcal{J}$ are σ^1 -closed, then $\mathcal{I} \cap \mathcal{J}$ is the σ^1-closure of $\mathcal{I}\mathcal{J}$.

$\mathrm{Lip}^2[a,b] = \{f \in \mathrm{Lip}[a,b] : f' \in \mathrm{Lip}[a,b]\} = \{f \in C^1[a,b] : f' \in \mathrm{Lip}[a,b]\}$

a Banach *-algebra with norm $\|f\|_{Lip^2} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$.

$\phi : \mathcal{A}_S^2 \to \mathcal{M}_S \bigoplus \mathcal{M}_S \bigoplus \mathcal{B}(\mathcal{H}), \phi(A) = (A, \delta_S(A), \delta_S^2(A))$.

The operator $\delta_S^2 : \mathcal{A}_S^1 \to \mathcal{B}(\mathcal{H})$, $\delta_S^2(A) = \delta_S(\delta_S(A))$ with domain $dom(\delta_S^2) = \mathcal{A}_S^2$.

$\mathcal{A}_S^2$ is ultra weakly dense in $\mathcal{M}_S$; and is σ^1-dense in $\mathcal{A}_S^1$.

Theorem

(1) The graph of the operator $\delta_S^2 : \mathcal{A}_S^1 \to \mathcal{B}(\mathcal{H})$ *given by* $G(\delta_S^2) = \{(A, \delta_S^2(A)) : A \in \mathcal{A}_S^2\}$ *is closed in* $\mathcal{A}_S^1 \bigoplus \mathcal{B}(\mathcal{H})$ *where* $\mathcal{A}_S^1$ *carries the* σ^1 *-topology and* $\mathcal{B}(\mathcal{H})$ *carries the ultra weak topology; and range of* ϕ *is an ultra weakly closed sub space of* $\mathcal{M}_S \bigoplus \mathcal{M}_S \bigoplus \mathcal{B}(\mathcal{H})$ *with the product ultra weak topology.*

(2)The algebra $\mathcal{A}_S^2$ *is dual of a Banach space, and the weak* **-topology on* $\mathcal{A}_S^2$ *denoted by* σ^2 *is given as* $A_\alpha \to A$ *in* σ^2 *if and only if* $A_\alpha \to A$ *ultra weakly,* $\delta_S(A_\alpha) \to \delta_S(A)$ *ultra weakly and* $\delta_S^2(A_\alpha) \to \delta_S^2(A)$ *ultra weakly.*

(3)Let $X = X^* \in \mathcal{A}_S^2$*. Let* $f \in Lip^2(sp(X))$*.Let* X *commutes with* $\delta_S(X)$*. Then* $f(X) \in \mathcal{A}_S^2$ *and*
$\|\delta_S^2(X)\| \leq L(f)\|\delta_S^2(X)\| + L(f')\|(\delta_S(X))^2\|$.

(4) Let $\mathcal{J}$ *be a* σ^2*-closed* **-ideal of* $\mathcal{A}_S^2$*. Then* $\mathcal{J}$ $=\sigma^2$*-closure of* $\mathcal{J}_S^2$.

Theorem

The operator S is self adjoint iff $\mathcal{F}_S^2$ has bounded approximate identity. In this case, $\mathcal{F}_S^2 = \mathcal{J}_S^2$.

$D(S^2)$ = a Hilbert space,

$< x, y >= (x, y) + (Sx, Sy) + (S^2x, S^2y)$

For $K \subset D(S^2)$,

$I_l(K)$ =closure in $<, >$ of span $\{x \bigotimes y : x \in K, y \in D(S^2)\}$

For a left ideal I of $\mathcal{F}_S^2$,

$L(I) = \{x \in D(S^2) : x \bigotimes y \in I$ for all $y \in D(S^2)\}$.

Theorem

The map $I \to L(I)$ gives a one one correspondance from non trivial closed essential left ideals of $\mathcal{F}_S^2$ to non trivial closed sub spaces of $D(S^2)$; and its inverse is $K \to I_l(K)$.

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