Maps between Grassmann manifolds

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The philosophy of algebraic topology is to associate, in a ‘natural’ manner, algebraic objects—which could be groups, vector spaces, rings, or just numerical invariants—to topological objects—spaces, maps, etc. Such association an leads to topological invariants.
The philosophy of algebraic topology is to associate, in a ‘natural’ manner, *algebraic* objects—which could be groups, vector spaces, rings, or just numerical invariants—to topological objects—spaces, maps, etc. Such association an leads to *topological invariants*.

This association helps convert the topological or geometric problem into an algebraic problem which is supposedly easier. A *negative* solution to the algebraic problem then implies a negative solution to the topological problem. But it is not always the case that a *positive solution* leads to a solution to the topological problem since there is usually loss of information while passing from topological to the algebraic realm.
An important problem in topology is the classification problem. Classification of all manifolds (or maps between them) is an impossible task. The coarser, homotopical classification, is relatively easier—but only relatively! Homotopy is, roughly speaking, the study of properties of spaces and maps invariant under continuous deformations.
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Denote by \([X, Y]\) the set of all maps \(f : X \rightarrow Y\) up to homotopy.
The cohomology algebra

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An important homotopy invariant of a space is its **cohomology algebra** $H^*(X; R) = \bigoplus_{k \geq 0} H^k(X; R)$ where $R$ is a fixed ring (usually $\mathbb{Z}$ or a field). To each $f : X \to Y$ there is associated an induced homomorphism of $R$-algebras $H^*(f; R) : H^*(Y; R) \to H^*(X; R)$. When $X = Y = S^n$, the $n$-dimensional sphere, this is a bijection. But this is a very rare occurrence. When $f$ is a constant map, $H^*(f; R)$ is zero in positive dimensions.
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This way we get a map $H^*(- - ; R) : [X, Y] \to \{ R\text{-algebra maps } H^*(Y ; R) \to H^*(X ; R) \}$ defined as $f \mapsto H^*(f ; R)$. 


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Let $f : M \to N$ be a map between two compact connected oriented manifolds of the same dimension, $d > 0$.

The **Brouwer degree** of $f$ is a numerical homotopy invariant which is much weaker than $H^*(f; \mathbb{Z})$. It is counts the (algebraic) number of times $f$ wraps $M$ around $N$. For example, degree of $z \mapsto z^n$ is $n$ when viewed as a map the circle $S^1$ or as a map of the Riemann sphere $S^2$.

When $\deg(f)$ is non-zero, then $H^*(f; \mathbb{Q})$ is a monomorphism. (The converse is trivially valid.)

Existence of degree 1 map yields a partial order, at least on the set of homotopy classes of all simply-connected manifolds of a fixed dimension.
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What is a Grassmann manifold?

The set of all \( k \)-dimensional vector subspaces of \( \mathbb{R}^n \), denoted \( \mathbb{R}G_{n,k} \) has the structure of a smooth compact manifold of dimension \( k(n - k) \). The complex Grassmann manifold \( \mathbb{C}G_{n,k} \) is defined similarly.
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An important special case of $FG_{n,k}$ is the projective space $FP^{n-1}$ (setting $k = 1$). Also, $FG_{n,k} \cong FG_{n,n-k}$. 
Theorem (J. Korbaš & — 1991) Let $n > k + 2l$, $1 \leq l < k \leq \lfloor n/2 \rfloor$. Then, for any $f : \mathbb{R}G_{n,k} \to \mathbb{R}G_{n,l}$, the induced map $H^*(f; \mathbb{Z}/2\mathbb{Z})$ is zero in positive dimensions, unless $n = 2^s$ or $(n, k) = (2^s - 1, 2)$, in which case the image is in the subalgebra generated by $H^1(G_{n,k}; \mathbb{Z}/2\mathbb{Z})$. 
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This is an analogue of the classical Borsuk-Ulam type theorem. It says that most of the time \( f \) is mod 2-homologically indistinguishable from a constant map.

Proof involves action of the Steenrod algebra on the mod 2 cohomology algebra of the Grassmann manifolds.
An application

**Theorem** (--- & K. K. Mukherjea, 1996) Assume that \( n > k + 2l - 1, n \neq 2^s \) and \( (n, k) \neq (2^s - 1, 2) \). Then given any map \( f : \mathbb{R}G_{n,k} \rightarrow \mathbb{R}G_{n,l} \), there exists a \( D \in \mathbb{R}G_{n,k} \) such that \( f(D) \subset D \).
Denote by $\tilde{G}_{n,k}$ the universal (double) cover of $\mathbb{R}G_{n,k}$, $n > 2$. $\tilde{G}_{n,k}$ is the space all oriented $k$-dimensional vector spaces in $\mathbb{R}^n$. 
Theorem (V. Ramani & —; 1997) Let $f : \tilde{G}_{n,k} \rightarrow \tilde{G}_{m,l}$ be any continuous map between two manifolds of the same dimension. Thus $k(n - k) = l(m - l)$. Assume that $k \leq n/2, 2 \leq l \leq m/2$ and $(n, k) \neq (m, l)$. Then the degree of $f$ is zero.
Theorem (V. Ramani & —; 1997) Let $1 < l < k \leq n/2$, $k(n - k) = l(m - l)$. Then, for any continuous map $f : \mathbb{C}G_{m,l} \rightarrow \mathbb{C}G_{n,k}$, degree of $f$ equals zero.
Theorem (— & S. Sarkar; 2007) Let $1 < l < k \leq n/2$, $k(n - k) = l(m - l)$ and $f : \mathbb{C}G_{n,k} \rightarrow \mathbb{C}G_{m,l}$.

(i) The degree of $f$ is not equal to $\pm 1$.
(ii) If $\deg(f)$ is non-zero, then $H^*(f)$ is determined.
(iii) Degree of $f$ is zero unless $Q = (k^2 - 1)(l^2 - 1)((n - k)^2 - 1)((m - l)^2 - 1)$ is a perfect square. In particular, for fixed $k$, there are only finitely many values of $n, m, l$ with the possibility that $\deg(f)$ is nonzero.

Proof of this above theorem uses Hodge theory, Schubert calculus, some standard algebraic topology, and a result in number theory due to Siegel. This is part of Swagata Sarkar’s thesis.
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Concluding remarks

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Regarding self-maps of (complex) Grassmann manifolds the following results are well-known:

- (E. Friedlander, 1975) There are continuous self-maps of complex Grassmann manifolds having arbitrarily large degree. (This is in sharp contrast to the Paranjape-Srinivas’ result in the realm of algebraic geometry!)

- (M. Hoffman, 1985) Any endomorphism, having non-zero degree, of the cohomology algebra $H^*(CG; R)$ is grading homomorphism, or, when $n = 2^k$, is the composition of grading with the canonical involution.

It is an unsolved problem whether vanishing of the degree of a self-map of a complex Grassmann manifold implies the vanishing in positive dimension of the induced map in cohomology.
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THANK YOU!