Maps between Grassmann manifolds

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This association helps convert the topological or geometric problem into an algebraic problem which is supposedly easier. A *negative* solution to the algebraic problem then implies a negative solution to the topological problem. But it is not always the case that a *positive solution* leads to a solution to the topological problem since there is usually loss of information while passing from topological to the algebraic realm.

Homotopy classification

An important problem in topology is the classification problem.
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When f is a constant map, $H^*(f;R)$ is zero in positive dimensions.

Let $f : M \longrightarrow N$ be a map between two compact connected **oriented** manifolds of the same dimension, $d > 0$.
The **Brouwer degree** of f is a numerical homotopy invariant which is much weaker than $H^*(f;\mathbb{Z})$. It is counts the (algebraic) number of times f wraps M around N.

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When $\deg(f)$ is non-zero, then $H^*(f;\mathbb{Q})$ is a monomorphism. (The converse is trivially valid.)

Existence of degree 1 map is yields a **partial order**, at least on the set of homotopy classes of all *simply-connected* manifolds of a fixed dimension.

What is a Grassmann manifold?

The set of all k-dimensional vector subspaces of $\mathbb{R}^n$, denoted $\mathbb{R}\mathbb{G}_{n,k}$ has the structure of a smooth compact manifold of dimension $k(n-k)$.
The complex Grassmann manifold $\mathbb{C}\mathbb{G}_{n,k}$ is defined similarly.

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They have very rich structures. Complex Grassmann manifolds admit certain additional structures–symplectic structure, Kähler structure, structure of a smooth projective variety.

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An important special case of $\mathbb{F}\mathbb{G}_{n,k}$ is the projective space $\mathbb{F}\mathbb{P}^{n-1}$ (setting $k=1$). Also, $\mathbb{F}\mathbb{G}_{n,k} \overset{\perp}{\cong} \mathbb{F}\mathbb{G}_{n,n-k}$.

Theorem (J. Korbaš & — 1991) Let $n > k + 2l$, $1 \leq l < k \leq \lfloor n/2 \rfloor$. Then, for any $f : \mathbb{RG}_{n,k} \longrightarrow \mathbb{RG}_{n,l}$, the induced map $H^*(f; \mathbb{Z}/2\mathbb{Z})$ is **zero in positive dimensions**, unless $n = 2^s$ or $(n, k) = (2^s - 1, 2)$, in which case the image is in the subalgebra generated by $H^1(G_{n,k}; \mathbb{Z}/2\mathbb{Z})$.

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This is an analogue of the classical Borsuk-Ulam type theorem. It says that most of the time f is mod 2-homologically indistinguishable from a *constant map.*
Proof involves action of the Steenrod algebra on the mod 2 cohomology algebra of the Grassmann manifolds.

Theorem (— & K. K. Mukherjea, 1996) Assume that $n > k + 2l - 1$, $n \neq 2^s$ and $(n, k) \neq (2^s - 1, 2)$. Then given any map $f : \mathbb{RG}_{n,k} \longrightarrow \mathbb{RG}_{n,l}$, there exists a $D \in \mathbb{RG}_{n,k}$ such that $f(D) \subset D$.

Denote by $\widetilde{G}_{n,k}$ the universal (double) cover of $\mathbb{R}\mathbb{G}_{n,k}$, $n > 2$. $\widetilde{G}_{n,k}$ is the space all **oriented** k-dimensional vector spaces in $\mathbb{R}^n$.

Theorem (V. Ramani & —; 1997) Let $f : \widetilde{G}_{n,k} \longrightarrow \widetilde{G}_{m,l}$ be any continuous map between two manifolds of the *same* dimension. Thus $k(n-k) = l(m-l)$. Assume that $k \leq n/2, 2 \leq l \leq m/2$ and $(n,k) \neq (m,l)$. Then the degree of f is zero.

Theorem (V. Ramani & —; 1997) Let $1 < l < k \leq n/2$, $k(n-k) = l(m-l)$. Then, for any continuous map $f : \mathbb{CG}_{m,l} \longrightarrow \mathbb{CG}_{n,k}$, degree of f equals zero.

Theorem (— & S. Sarkar; 2007) Let $1 < l < k \leq n/2$, $k(n-k) = l(m-l)$ and $f : \mathbb{C}\mathbb{G}_{n,k} \longrightarrow \mathbb{C}\mathbb{G}_{m,l}$.

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(iii) Degree of f is zero unless $Q = (k^2-1)(l^2-1)((n-k)^2-1)((m-1)^2-1)$ is a perfect square. In particular, for fixed k, there are only finitely many values of n, m, l with the possibility that $deg(f)$ is nonzero.

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Proof of this above theorem uses Hodge theory, Schubert calculus, some standard algebraic topology, and a result in number theory due to Siegel. This is part of Swagata Sarkar's thesis.

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(M. Hoffman, 1985) Any endomorphism, having non-zero degree, of the cohomology algebra $H^*(\mathbb{C}\mathbb{G};\mathbb{R})$ is grading homomorphism, or, when $n = 2k$, is the composition of grading with the canonical involution.

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It is an unsolved problem whether vanishing of the degree of a self-map of a complex Grassmann manifold implies the vanishing in positive dimension of the induced map in cohomology.

THANK YOU!