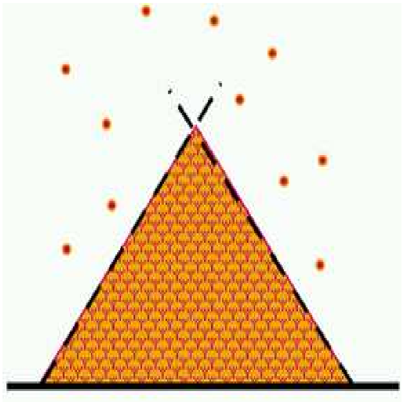
Abelian Sandpile Model (ASM)
and
Infinite Volume Limit

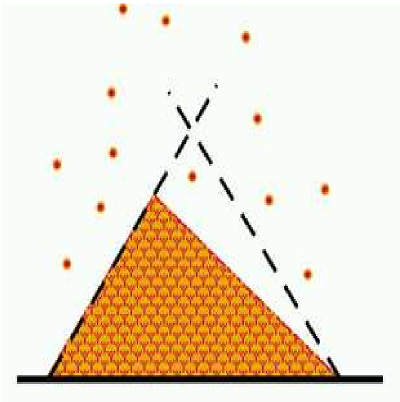
Infinite volume limit for the stationary distribution of Abelian sandpile models.
http://www.isid.ac.in/∼athreya/Research

Joint work with Antal Jarai.

To Appear in Communications in Mathematical Physics

Sandpile Images





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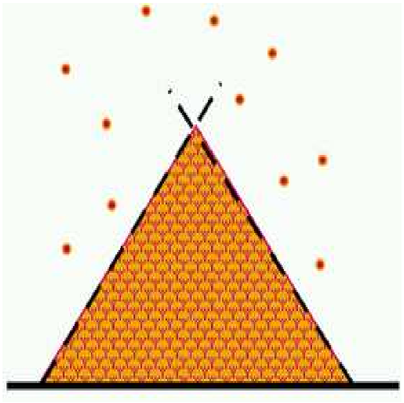

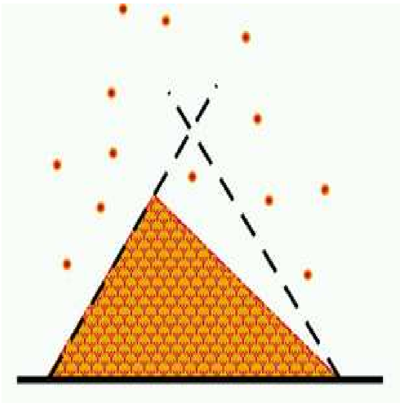

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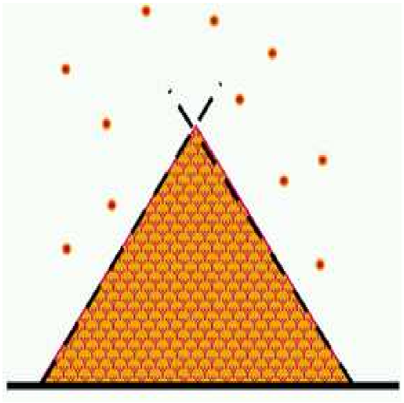

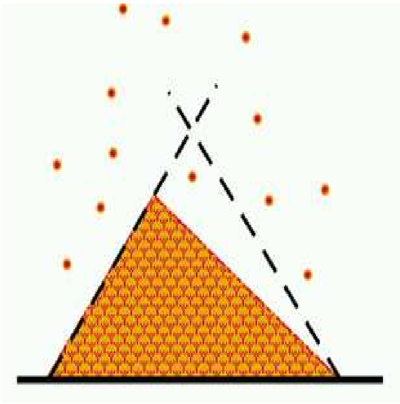

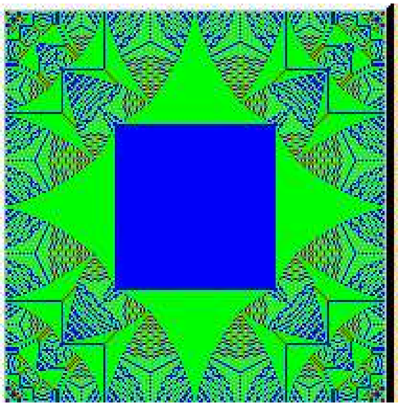

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- ASM is a simple model where SOC can be studied.
- Other SOC: Forest Fire models.

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- Repeat Step 1.

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$1\ 2\ 2\ 2$

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1 2 2 2

add a sand grain

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1 2 3 2

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Dynamics

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  - makes the model tractable and easy to analyze
  - Due to the random choice of i, we have a Markov-chain with state space Ω_Λ.
    - There is a unique stationary distribution ν_Λ, which is uniform on the set of recurrent states of the Markov chain [Dhar 90].

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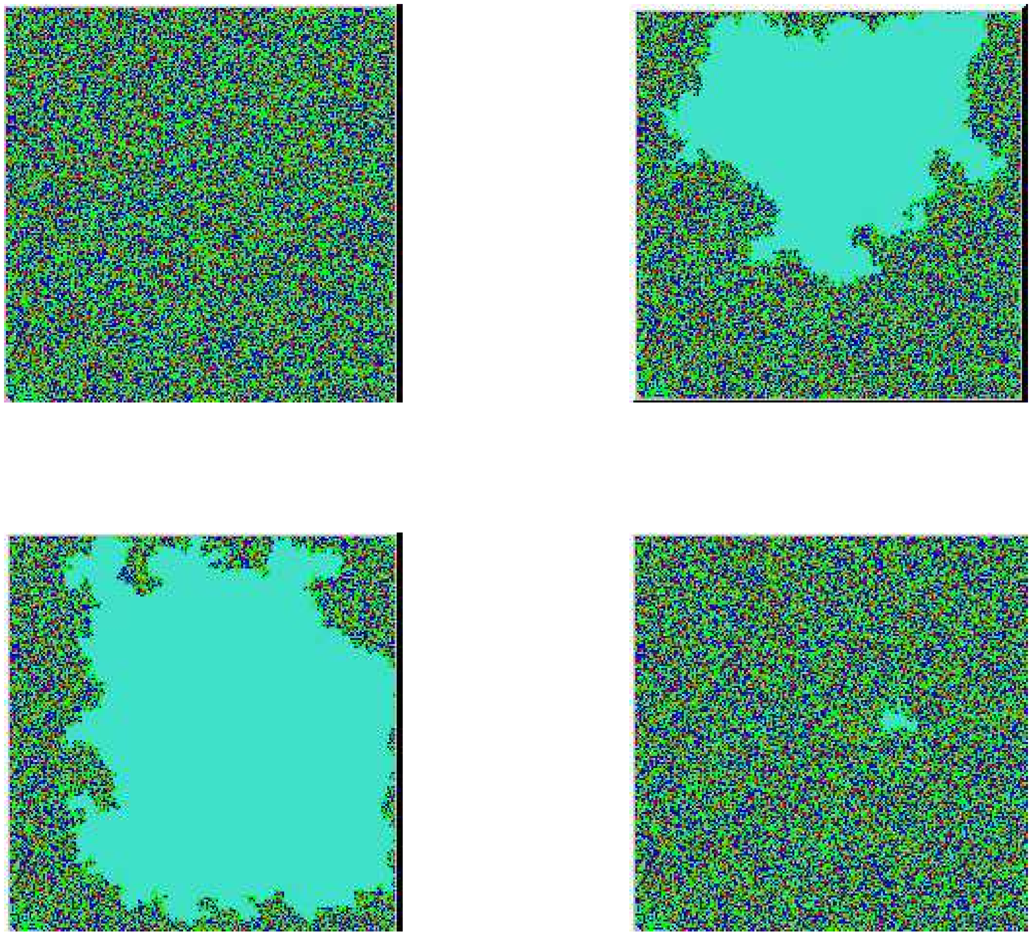
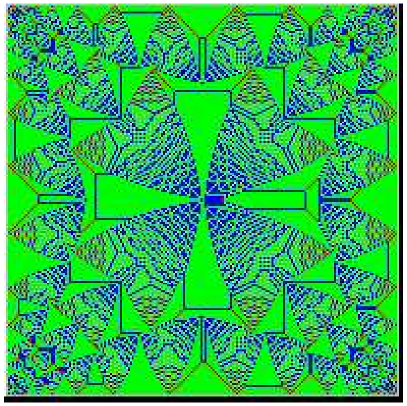
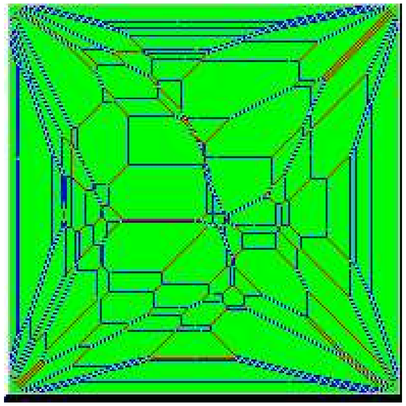
  It is often assumed that these quantities have distributions with a power law tail in the limit $\Lambda \nearrow \mathbb{Z}^d$.

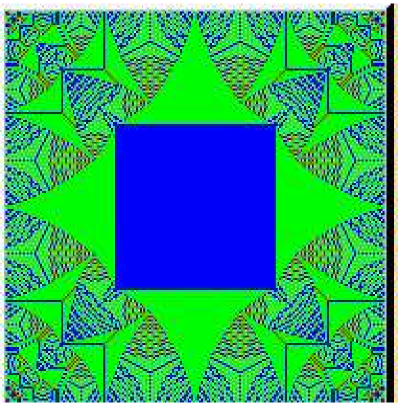
ASM- Range of Avalanche



Source: http://thy.phy.bnl.gov/www/xtoys.html

ASM- Recurrent and Identity State



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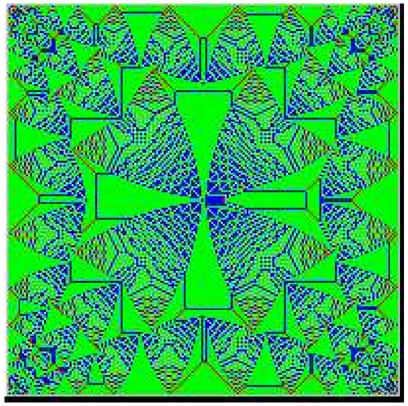

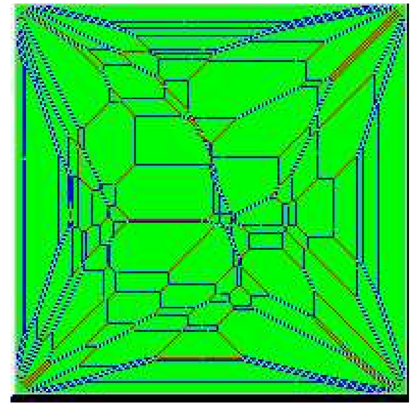

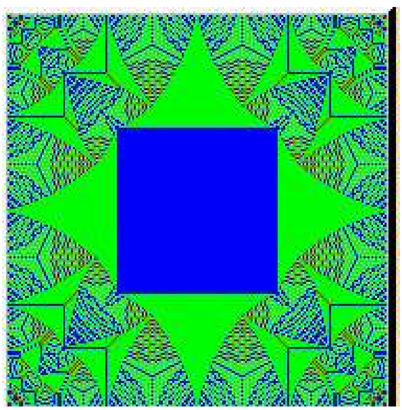

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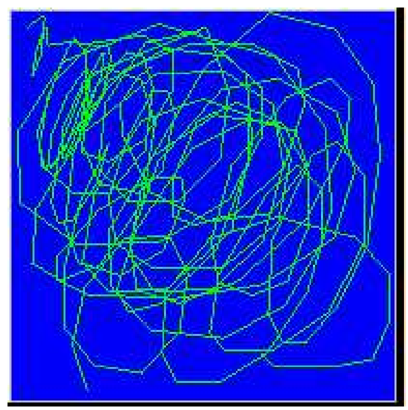

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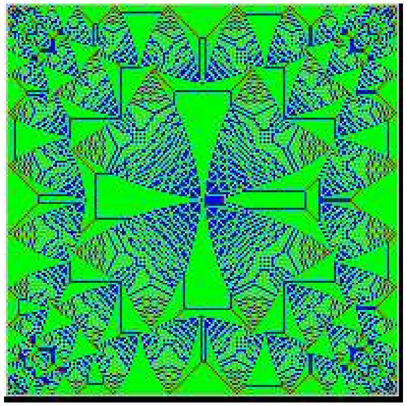

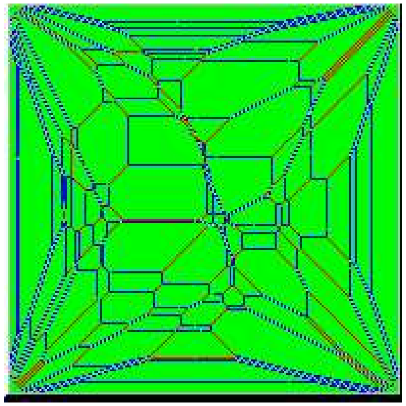

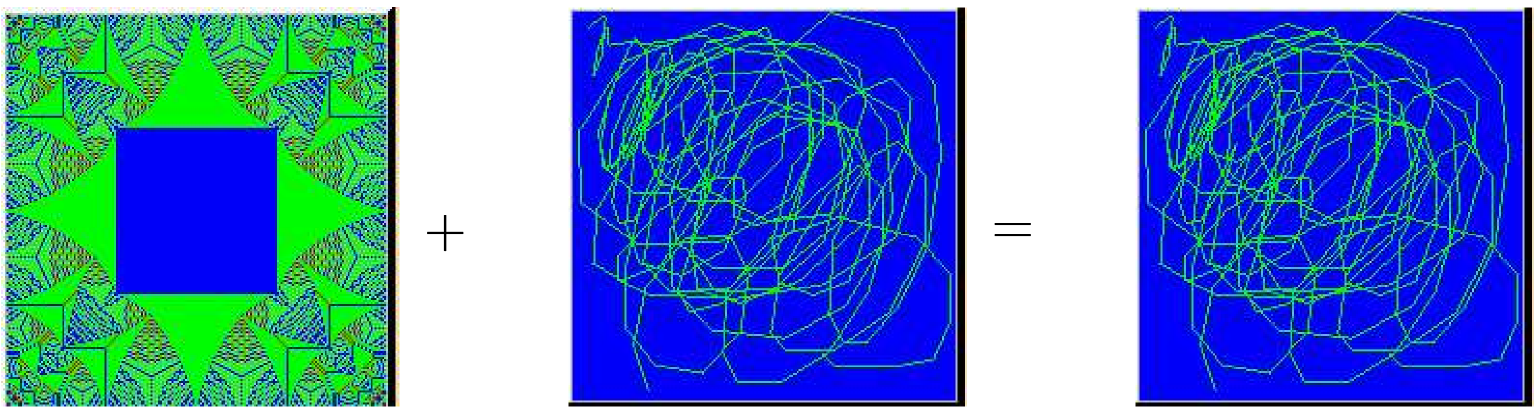

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- To the best of our knowledge, there is no rigorous proof of power law behavior, either in $d = 2$ or higher.
  - As a step in analyzing the above distributions, we consider the limit $\Lambda \nearrow \mathbb{Z}^d$, and define avalanche characteristics in the infinite volume.

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- Let S be the quantity of interest, say size of an avalanche. There is a measure ν on $\Omega = \{1, \ldots, 2d\}^{\mathbb{Z}^d}$ such that

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- From computations in [MD 91]: Under ν, at least the random field $I[z_x = 1]$ has power law correlations.

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Theorem 1. *Let $d \geq 2$. The measures ν_n weakly converge to a translation invariant measure ν on Ω. For any cylinder event E and any $v \in \mathbb{Z}^d$ we have*

$$\nu(E) = \lim_{n\to\infty} \nu_n(E) = \lim_{n\to\infty} \nu_n(\tau_v E)$$

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- We give expressions for $\nu(E)$ in terms of the USF on $\mathbb{Z}^d$.

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Thank you for your attention