Tertiary classes–after Chern-Simons theory

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Overview

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- Euler characteristic class.
- Vector bundles and universel characteristic classes.
- Connections and Chern-Weil theory–primary classes.
- Flat connection and Chern-Simons theory–secondary classes.
- One-parameter variation of flat connections–tertiary classes.

Euler characteristic class

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- Characteristic classes are global invariants which measure deviation of a local product structure from a product structure. They are closely related to "curvature" in differential geometry.
- A finite cell complex M looks like a finite disjoint union of open cells of varying dimension. For instance, a sphere is disjoint union of 2-dimensional open ball/cell and a 0-dim'al cell- a point.
- Euler characteristic class:

$$\chi(M) := \sum_k (-1)^k \alpha_k$$

Here α_k is the number of k-dim'al cells.
• Example: If M is a two-dim'al polyhedron with $v =$ no. of vertices, $e =$ no. of edges and $f =$ no. of faces, then

$$\chi(M) = v - e + f.$$

Whitney explored this notion further for smooth manifolds (locally they look like open subsets of R^n), and realised $\chi(M)$ as the number of zeroes of a smooth vector field on M.

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- There is a universel vector bundle of rank r:

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- Fix a flag of subspaces:

$$V_N \subset V_{N+1} \subset ... \subset V_{r+N} = \mathbb{C}^{r+N}.$$

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- Define universel "Chern classes":

$$c_i := \{[W] \in G(r, r+N) : \dim(W \cap V_{i+N-1}) \geq i, 1 \leq i \leq r\}.$$

- Whitney-Pontryagin embeddding theorem: A vector bundle $E \to M$ (i.e. a bundle of cx vector spaces of rank r over M) is induced by a continuous map:

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- A **'connection'** ∇ on E is a formula to differentiate vector valued functions on M. More precisely, smooth functions $s : U \to U \times \mathbb{C} \subset E$,where $U \subset M$ is open and $s(x) = (x, v_x)$.

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- Chern-Weil theory defines invariants/forms
$$c_i(E, \nabla) \in H^{2i}(M, \mathbb{C}),\ 1 \leq i \leq r.$$

- The **'curvature'** form is $\Theta := \nabla \circ \nabla$.

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- A canonical lifting is provided by Chern-Cheeger-Simons theory (by introducing 'Differential cohomology'):

$$\hat{c}_i(E, \nabla) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Z}), \; 0 \leq i \leq r.$$

These are the Chern-Simons classes/secondary classes of (E, ∇).

Variation of flat connections–tertiary classes

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- Suppose $\gamma := \{\nabla_t\}_{t\in[0,1]}$ is a one-parameter family of flat connections. Then the rigidity formula says:

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- We obtain a canonical lifting

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  and show they depend only on the homotopy class of γ.

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- This gives a homomorphism:

$$\pi_1(\text{moduli space of flat connections}) \to \oplus_{i\geq 3} H^{2i-2}(M,\mathbb{C}/\mathbb{Z}).$$

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