# Morphism of Varieties

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# Introduction

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When two objects are given one would like to compare them and try to see wether all the features of one object is contained in the other.

In mathematics objects are sets with some special properties. The comparison takes place via maps between the objects. In different branches of mathematics different kind of maps are considered.

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In the branch named Algebraic Geometry an object is a set defined locally by polynomials. The maps are set maps which carry subsets which are defined by polynomials to subsets defined by polynomials.

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**The aim of this talk is to introduce the objects in Algebraic Geometry known as Varieties, which are locally defined by polynomials, and the maps between varieties which preserve these local structures.**

# Contents

# Polynomials

- **A monomial in n-variables $X_1, \ldots, X_n.$, over the field $\mathbb{C}$ of complex numbers, is an expression of the form**

$$a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n},$$

**where the coefficient $a_{i_1,\ldots,i_n} \in \mathbb{C}$ and $i_1, \ldots, i_n$ are non negative integers.**

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- **The non negative integer $i_1 + \cdots + i_n$ is called the degree of the monomial.**
- **For example $a \in \mathbb{C}$ is a monomial of degree $0$.**

$$X_1 X_2, \;\; X_1^2$$

**are monomials of degree 2.**

$$12X_1X_2X_3, \;\; 15X_1^2X_3, \;\; 18X_3^3$$

**are monomials of degree 3.**

- **A polynomial is a finite sum of monomials and its degree is the greatest degree monomial appearing in it. A polynomial is said to be homogeneous, if it is sum of monomials of same degree.**

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- $X_1^2 + \cdots + X_n^2 - 1$ **is a degree 2 polynomial in** $n$ **variables.**
- $X_1^m + \cdots + X_n^m$ **is a homogeneous polynomial of degree** $m$ **in** $n$ **variables.**

# Contents

# Affine Varieties

- **Let** $\mathbb{C}[X_1, \ldots, X_n]$ **be the set of all polynomials in** $n$ **variables over** $\mathbb{C}$ **and** $f(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$.

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- **The substitution of entries of coordinates of a point of $\mathbb{C}^n$ in to variables of $f(X_1, \ldots, X_n)$ define a map $f : \mathbb{C}^n \to \mathbb{C}$ :**

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- **These maps are called polynomial maps from $\mathbb{C}^n$ to $\mathbb{C}$..**

- **Given** $f_i(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$ $i = 1, \ldots, r.$, **the set of common zeros**

  $$Z(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n | f_i(a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, r.\}$$

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- **Using polynomial maps we can define nearness of points of an Affine algebraic set.**

- **An Affine algebraic set** $Z(f_1, \ldots, f_s) \subset \mathbb{C}^n$ **is said be an Affine variety if it is not a union of two proper Affine algebraic subsets. In other words, if**

  $$Z(f_1, \ldots, f_s) = Z(h_1, \ldots, h_k) \cup Z(g_1, \ldots, g_l)$$

  **then either** $Z(f_1, \ldots, f_s) = Z(h_1, \ldots, h_k)$ **or** $Z(f_1, \ldots, f_s) = Z(g_1, \ldots, g_l)$.

- Consider $Z(X_1X_2) \subset \mathbb{C}^2$. Then $Z(X_1X_2) = Z(X_1) \cup Z(X_2)$ but $Z(X_1X_2) \neq Z(X_1)$ and $Z(X_1X_2) \neq Z(X_2)$, thus $Z(X_1X_2)$ is *not* an Affine variety.

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- **Following are some examples of Affine varieties:**
- $Z(a_1X_1 + a_2X_2) \subset \mathbb{C}^2$ —> **line through the origin in** $\mathbb{C}^2$.

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- $Z(a_1X_1 + a_2X_2) \subset \mathbb{C}^2$ —> **line through the origin in** $\mathbb{C}^2$.
- $Z(X_1X_2 - 1) \subset \mathbb{C}^2$ —> **complex hyperbola.**
- $Z(X_1^2 + X_2^2 + X_3^2 - 1) \subset \mathbb{C}^3$ —> **complex sphere.**

- **If $Z \subset \mathbb{C}^n$ is an Affine variety then by restricting the polynomial functions on $\mathbb{C}^n$ to $Z$ we get maps from $Z$ to $\mathbb{C}$.**

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- **If $X$ and $Y$ are Affine varieties, then a map $\phi : X \to Y$ is said to be a morphism of Affine varieties, if $\phi$ takes (by composition of maps) polynomial functions on $Y$ to polynomial functions on $X$.**

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# Varieties

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- **A Variety is obtained by glueing finitely many Affine varieties along suitable proper subsets.**
- **Example: The set $\mathbb{P}^n$ of all one dimensional subspaces of $\mathbb{C}^{n+1}$ has a variety structure. The subset $U_i \subset \mathbb{P}^n$, consists of one dimensional subspaces generated by vectors whose $i$th coordinate is non zero, can be identified with $\mathbb{C}^n$. Thus $\mathbb{P}^n$ is obtained by taking $n+1$ copies of $\mathbb{C}^n$ and identifying them.**

- **The variety $\mathbb{P}^n$ is called the Projective $n$ space. Given a homogeneous polynomial $F$ in $n+1$ variables the polynomial map defined by $F$ takes the value zero at $p \in \mathbb{C}^{n+1} - \{(0, \ldots, 0)\}$ then it is zero on the whole one dimensional subspace generated by $p$.**

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- **If $F_1,\ldots,F_s$ are homogeneous polynomials in $n+1$ variables,then the set of common zeros of $F_1,\ldots,F_s$ in $\mathbb{P}^n$ is denoted by**

$$Z(F_1,\ldots,F_s).$$

- **A subset of $\mathbb{P}^n$ of the form $Z(F_1, \dots, F_s)$ is called a Projective algebraic set. It is easy to see that any projective algebraic set is a finite union of of Affine algebraic sets. A Projective algebraic variety is defined in a similar way as an Affine algebraic varietiey.**

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- **Fermat curves $Z(X_1^m + X_2^m + X_3^m) \subset \mathbb{P}^2$ and the the Ellipic curves $Z(X_1^2 X_3 - X_2^3 + bX_3^3) \subset \mathbb{P}^2$ are some examples of Projective algebraic varieties.**

# Contents

# Morphism

- **Given a variety $Z$ and a point $p \in Z$ we can talk of polynomial functions near $p$. This is because $p \in U \subset Z$, with $U$ an Affine algebraic set. A local Regular function or a local Algebraic function near $p$ is a function near $p$ which has an expression $f/g$, where $f, g$ are polynomial functions near $p$, $g(p) \neq 0$ and $g(q) \neq 0$ at all points $q$ near $p$.**

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- **A map** $\phi : X \to Y$ **between varieties is called a Morphism of varieties, if** $\phi$ **takes local algebraic functions on** $Y$ **to local algebraic functions on** $X$.

- My current area of research is to study of morphisms of varieties: Given two varieties one tries to describe all possible morphisms between them, and study the properties of these morphisms.

Thank You.