Homogenization and Optimal Control

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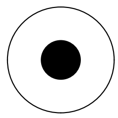
- Homogenization permits us to study the global behaviour of heterogeneous bodies with a lot of heterogeneities whose dimensions are small compared to those of the body.

- It describes the macroscopic behaviour of systems with a fine microstructure.

Example: Consider a beam of uniform cross-section as in the figure below.



The beam is made of two materials occupying the light and dark regions respectively. To study its torsional rigidity, we need to solve the following boundary value problem.

$$\begin{aligned} -\mathrm{div}(A(x)\nabla u(x)) &= 2, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω denotes the cross-section of the beam. The torsional rigidity is defined as

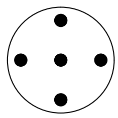
$$\int_\Omega |\nabla u(x)|^2 \, dx.$$

The matrix $A(x)$ is defined as follows:

$$A(x) = \begin{cases} \mu I, & \text{in the light region} \\ \nu I, & \text{in the dark region} \end{cases}$$

where μ and ν are positive constants depending on the elastic properties of the material the region represents and I denotes the identity matrix.

Now consider the beam with several inclusions.



Then the diagonal entries of the matrix $A(x)$ oscillate between the values μ and ν. This makes numerical computation very difficult if the number of such oscillations is very large. Homogenization seeks to describe the effective behaviour by replacing the heterogeneous material with a suitable (fictitious) homogeneous material.

The Mathematical Problem

- Let $\Omega \subset \mathbb{R}^N$ be a bounded domain.

- Let $\{A_\varepsilon(x)\}$ be a family of $N \times N$ matrices,

$$A_\varepsilon(x) \;=\; (a_{ij}^\varepsilon(x)).$$

  Assume that, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|\xi|^2 \;\leq\; \sum_{i,j=1}^{N} a_{ij}^\varepsilon(x)\xi_i\xi_j \;\leq\; \beta|\xi|^2$$

  for all $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbb{R}^N$, where $|\xi|^2 = \sum_{i=1}^N |\xi_i|^2$.

Consider the boundary value problem:

$$\begin{aligned} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) &= f, \quad \text{in } \Omega, \\ u_\varepsilon &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

For fixed f, it can be shown that u_ε converges (at least for a subsequence) to a function u in a suitable Hilbert space ($H_0^1(\Omega)$).

Questions:

- Is u the solution of a similar problem?
- If so, identify the differential operator.

Consider the one-dimensional case:
Let $0 < \alpha \le a_\varepsilon(x) \le \beta$. Consider the problem:

$$-\frac{d}{dx}\left(a_\varepsilon(x)\frac{du_\varepsilon}{dx}(x)\right) = f(x),\ 0 < x < 1,$$

with the boundary conditions

$$u_\varepsilon(0) = u_\varepsilon(1) = 0.$$

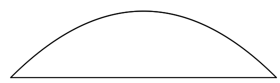
Since a_ε and $1/a_\varepsilon$ are bounded uniformly, they converge, in some (weak) topologies. Let $\overline{a}(x)$ be such that

$$\frac{1}{a_\varepsilon} \rightarrow \frac{1}{\overline{a}}.$$

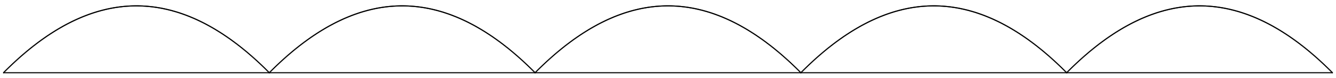
Then u_ε converges (in a suitable topology) to u which is the unique solution of the problem:

$$\begin{aligned} -\tfrac{d}{dx}\left(\overline{a}(x)\tfrac{du}{dx}(x)\right) &= f(x),\ 0 < x < 1, \\ u(0) = u(1) &= 0. \end{aligned}$$

The Periodic case Let $0 < \alpha \leq a(x) \leq \beta$ be a function defined on the interval $[0, 1]$.



Let $\varepsilon > 0$. On $[0, \varepsilon]$, define $a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ and extend it periodically, with period ε, throughout the real line, and continue to denote this function as a_ε.



Then

$$a_\varepsilon \ \rightarrow \underline{a} \ = \ \int_0^1 a(y) \ dy.$$

Similarly,

$$\frac{1}{a_\varepsilon} \ \rightarrow \ \frac{1}{\overline{a}} \ = \ \int_0^1 \frac{1}{a(y)} \ dy.$$

Note:

$$\overline{a} \ \neq \ \frac{1}{\underline{a}}.$$

Here, $\underline{a}$ is the *arithmetic* mean, while $\overline{a}$ is the *harmonic* mean.

Layered Material

$\Omega \subset \mathbb{R}^2$. Let $a(x)$ and $a_\varepsilon(x)$ be as before. Let

$$A_\varepsilon(x,y) = \begin{bmatrix} a_\varepsilon(x) & 0 \\ 0 & a_\varepsilon(x) \end{bmatrix}.$$

Then, the matrix A corresponding to the homogenized limit will be given by

$$A(x,y) = \begin{bmatrix} \left(\int_0^1 \frac{1}{a(s)}\, ds\right)^{-1} & 0 \\ 0 & \int_0^1 a(s)\, ds \end{bmatrix}.$$

Homogenization Techniques

- Periodic case
  - multiscale expansions (formal method)
  - energy method, or method of oscillating test functions
  - multiscale convergence
  - Bloch wave method
- H-convergence
- G-convergence
- Γ-convergence

If the A_ε are as assumed earlier, there exists A such that whenever u_ε solves

$$(P_\varepsilon)\begin{cases} -\text{div}(A_\varepsilon \nabla u_\varepsilon) = f, & \text{in } \Omega \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

we have

$$u_\varepsilon \rightarrow u$$

'weakly in $H_0^1(\Omega)$' where u solves the 'homogenized problem':

$$(P)\begin{cases} -\text{div}(A \nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The matrix A is called the homogenized or H-limit of $\{A_\varepsilon\}$.

We also have the convergence of the ‘energy’:

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \; dx \; \to \int_\Omega A \nabla u . \nabla u \; dx.$$

However, because of the weak convergence,

$$\int_\Omega |\nabla u_\varepsilon|^2 \; dx \; \not\to \; \int_\Omega |\nabla u|^2 \; dx.$$

More generally, if $\{B_\varepsilon\}$ is another family of matricies with properties similar to those of $\{A_\varepsilon\}$, can we express the limit of

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \; dx$$

in terms of u?

The answer comes as a by-product of our analysis of an optimal control problem that we now describe.

- Let $\{A_\varepsilon(x)\}$ be a family of $N \times N$ matrices,

$$A_\varepsilon(x) \;=\; (a^\varepsilon_{ij}(x)).$$

Assume that, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|\xi|^2 \;\leq\; \sum_{i,j=1}^{N} a^\varepsilon_{ij}(x)\xi_i\xi_j \;\leq\; \beta|\xi|^2$$

for all $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbb{R}^N$, where $|\xi|^2 = \sum_{i=1}^{N} |\xi_i|^2$.

- Let $\{B_\varepsilon(x)\}$ be a family of $N \times N$ matrices,

$$B_\varepsilon(x) \;=\; (b^\varepsilon_{ij}(x)).$$

Assume that, there exist constants $\alpha_1 > 0$ and $\beta_1 > 0$ such that

$$\alpha_1|\xi|^2 \;\leq\; \sum_{i,j=1}^{N} b^\varepsilon_{ij}(x)\xi_i\xi_j \;\leq\; \beta_1|\xi|^2$$

for all $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbb{R}^N$, where $|\xi|^2 = \sum_{i=1}^{N} |\xi_i|^2$.

An Optimal Control Problem

- $L^2(\Omega)$ - Control Space.

- $U \subset L^2(\Omega)$, a closed, convex subset - the set of admissible controls.

- $f \in L^2(\Omega)$ - given.

- For $\theta \in U$, define $y_\varepsilon = y_\varepsilon(\theta) \in H_0^1(\Omega)$ as the unique solution of the problem:

$$\begin{aligned} -\text{div}(A_\varepsilon \nabla y_\varepsilon) &= f + \theta, \quad \text{in } \Omega \\ y_\varepsilon &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

- $y_\varepsilon = y_\varepsilon(\theta)$ is the state associated to the control θ.

Assume that the B_ε are all symmetric matrices.
Cost Function:

$$J_\varepsilon(\theta) \; = \; \int_\Omega B_\varepsilon \nabla y_\varepsilon . \nabla y_\varepsilon \; dx + \int_\Omega \theta^2 \; dx.$$

Optimal Control Problem:
Find $\theta_\varepsilon^* \in U$ such that

$$J_\varepsilon(\theta_\varepsilon^*) \; = \; \min_{\theta \in U} J_\varepsilon(\theta).$$

Facts:

- There exists a unique $\theta_\varepsilon^* \in U$ which minimizes J_ε.

- $\theta_\varepsilon^* \rightarrow \theta^*$ (weakly) in $L^2(\Omega)$.

Questions:

- Is θ^* the optimal control corresponding to a similar problem?
- If so, what is the state equation in the limit?
- What is the cost functional in the limit?

Answer is ‘Yes’.

- The state equation involves the ‘usual’ homogenized matrix A associated to the family $\{A_\varepsilon\}$.

- The cost functional involves a matrix $B^{\#}$ which is dependent on both $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$.

- $\theta^*_\varepsilon \to \theta^*$ in $L^2(\Omega)$ strongly.

- $$\int_\Omega B_\varepsilon \nabla y^*_\varepsilon . \nabla y^*_\varepsilon \; dx \to \int_\Omega B^{\#} \nabla y^* . \nabla y^* \; dx$$
  where y^*_ε is the optimal state corresponding to the optimal control θ^*_ε and the state equation involving A_ε and y^* is the state corresponding to θ^* and the state equation involving A.

One-dimensional case

Let

- $0 < \alpha \leq a_\varepsilon(x) \leq \beta$

- $0 < \alpha_1 \leq b_\varepsilon(x) \leq \beta_1$

- $U \subset L^2(0,1)$

- State equation:

$$\begin{aligned} -\frac{d}{dx}\left(a_\varepsilon \frac{dy_\varepsilon}{dx}\right) &= f + \theta, \ 0 < x < 1 \\ y_\varepsilon(0) = y_\varepsilon(1) &= 0. \end{aligned}$$

- Cost Functional:

$$J_\varepsilon(\theta) = \int_0^1 b_\varepsilon \left(\frac{dy_\varepsilon}{dx}\right)^2 \, dx + \int_0^1 \theta^2 \, dx.$$

Limit Problem

- Let
$$\frac{1}{a_\varepsilon} \longrightarrow \frac{1}{\overline{a}}.$$

- Let
$$\frac{1}{g_\varepsilon} \stackrel{\text{def}}{=} \frac{b_\varepsilon}{a_\varepsilon^2} \longrightarrow \frac{1}{\overline{g}}.$$

- State equation:
$$\begin{aligned} -\tfrac{d}{dx}\left(\overline{a}\tfrac{dy}{dx}\right) &= f+\theta,\ 0<x<1, \\ y(0)=y(1) &= 0. \end{aligned}$$

- Cost Functional:
$$J(\theta) = \int_0^1 b^{\#}\left(\frac{dy}{dx}\right)^2\ dx + \int_0^1 \theta^2\ dx$$
where
$$b^{\#} = \frac{\overline{a}^2}{\overline{g}}.$$

Answer to an earlier question

Let a_ε be a scaled periodic extension of a function a as described earlier. Let $f \in L^2(0,1)$ be given. Consider the problem:

$$\begin{aligned} -\frac{d}{dx}\left(a_\varepsilon \frac{du_\varepsilon}{dx}\right) &= f,\ 0 < x < 1 \\ u_\varepsilon(0) = u_\varepsilon(1) &= 0. \end{aligned}$$

We can show the following:

$$\lim_{\varepsilon \to 0} \int_0^1 \left(\frac{du_\varepsilon}{dx}\right)^2 \ dx \ = \ \int_0^1 b^{\#} \left(\frac{du}{dx}\right)^2 \ dx$$

where

- u satisfies:

$$\begin{aligned} -\frac{d}{dx}\left(\overline{a}\frac{du}{dx}\right) &= f,\ 0 < x < 1 \\ u(0) = u(1) &= 0, \end{aligned}$$

- $$\overline{a} = \left(\int_0^1 \frac{1}{a(s)}\, ds\right)^{-1},$$

- $$b^{\#} = \frac{\overline{a}^2}{\overline{g}}$$

  and

$$\overline{g} = \left(\int_0^1 \frac{1}{a(s)^2}\, ds\right)^{-1}.$$