Isotropy of quadratic forms

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i.e. if a, b, c are three integers such that $a^n + b^n = c^n$ and $n \geq 3$, then one of the a, b, c must be zero.

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n is called the *dimension* of q

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Witt, in the early 30's introduced a new direction in the study of quadratic forms — now termed as *algebraic theory of quadratic forms* — by introducing the Witt group of quadratic forms to study the totality of quadratic forms over a general field k.

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More generally $X_1^2 + X_2^2 + \cdots + X_n^2$ is an anisotropic quadratic form over $\mathbb{R}$

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Since $u(k) = 4$, it is easy to see that $u(k(T)) \geq 8$

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Theorem(Parimala - Suresh)(1998): Let k be a p-adic field. If $p \neq 2$, then $u(k(X)) \leq 10$.

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Theorem(Parimala - Suresh)(2007) : Let k be a p-adic field. If $p \neq 2$, then $u(k(X)) = 8$.

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The proof of the above theorems uses many techniques from cohomology, arithematic geometry and Bruaer groups.