On Spectrum and Arithmetic

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The concept of spectrum arose from Newton's theory of light explaining the nature of different colors.
White light scatters to give different colors.
Light is a wave given by a function of the form

$$f(x) = Asin(vx) + bcos(vx), \quad v\text{positive.}$$

Colors correspond to different values of v.
These functions are special 'harmonic' functions; they are eigenfunctions of the Laplace operator:

$$\Delta = -d^2/dx^2,$$

with eigenvalues v^2:

$$\Delta f = v^2 f$$

Fourier transform says that any 'nice' function can be decomposed as a sum of harmonic functions.

Harmonic functions or eigenfunctions of the Laplacian and their eigenvalues also arise in the theory of sound, for example the motion of a vibrating string.
Here the equation for the harmonics with total string length of one is given by

$$f(x) = Asin(2\pi nx)$$

where n runs over integers. These functions are eigenfunctions of the Laplacian with eigenvalue $4\pi^2 n^2$.

- different eigenvalues correspond to different harmonics or different sounds.
- the harmonics are 'quantized', the eigenvalues form a discrete set.

More generally take a drum- not necessarily circular but of different shapes.

A given drum will give raise to different sounds.

A catchy formulation of a basic question is the following:

Can you hear the shape of a drum?

In other words, knowing the sounds that emanate from the drum can you figure out the shape of the drum?

Answer: NO.

Basic purport of my talk: One can hear the arithmetic of a ‘class of drums’ and conversely.

Mathematically, these are compact locally symmetric spaces arising from congruent arithmetic lattices.

In the rest of the talk I will try to give a flavor of the various concepts involved and what we can say about it.

First of all there are more general contexts in which we can define Laplacians. This is in the context of Riemannian Geometry.
A manifold is a subspace of n-dimensional space which is smooth in the neighbourhood of each point.
In particular it has a well defined tangent space at each point.
Examples: surface of a ball, the surface of a medu vada or a tennikoit ring.
Sharp corners, intersecting lines, cusps are not allowed.
The concept of a manifold allows us to carry over concepts of calculus to more general spaces. So one can differentiate functions, etc.

To talk about lengths, distances and angles you need the concept of a Riemannian manifold.
Based on the observation that the square of the lengths of vectors is a nice algebraic function: it is a homogenous quadratic (scaling a vector increases the length by a square of the scaling) polynomial meaning of degree two. These are called quadratic forms.
A Riemannian manifold (M, g) consists of the following:

- a smooth manifold M.
- at each point p of the manifold a quadratic form g_p defined on the tangent space.
- This quadratic form should vary smoothly as p varies.

Symmetric spaces: These are spaces with a large group of isometries.

- They are homogenous: given any two points on the manifold there is an isometry taking one point to the other point.
- the isometry group is a semi-simple Lie group.
- The group of isometries fixing a point is a maximal compact subgroup.

Examples: Standard Euclidean n-space.
Spheres.
Hyerbolic spaces: Consider the group $SL(2, R)$: this consist of 2 by 2 real matrices of determinant one.
The group $SL(2, R)$ acts on the hyperbolic upper half plane by isometries.

If the manifold is sitting inside n-dimensional space, then we get for free a Riemannian metric, by restricting the standard quadratic form given by taking squares of lengths.
But it is useful to take a more abstract viewpoint.
Examples: linear subspaces and circles (flat geometry), spheres (positively curved or elliptic geometry).
Hyperbolic upper half plane: consider the open half space with the y-co-ordinate positive.
For a tangent vector v based at the point (x, y) of length square $|v|^2$, define the hyperbolic length to be:

$$g_{x,y}(v) = |v|^2/y^2.$$

i.e., scale the length by the inverse square of the distance from the x-axis.
Two Riemannian manifolds are said to be isometric if there is a one to one invertible correspondence between them preserving the metric structures (the correspondence is required to be smooth to start with).

On a Riemannian manifold can define the analogue of the classical Laplace operator Δ acting on smooth functions defined on the manifold. This has the following properties:

- a second order differential operator. It is natural in that it is invariant under local isometries.
- The leading term (the second degree term) is given by the metric. In other words it is the lift of the metric to a natural second order differential operator.
- self-adjoint.
- It is elliptic.

Assume the manifold is compact, i.e., closed and bounded. Then the Laplacian can be diagonalized: there is a basis of eigenfunctions

$$\Delta(f) = e(f)f.$$

The eigenvalues $e(f)$ form a discrete subset of the real numbers.
Define the spectrum $Spec(M, g)$ to be the collection of eigenvalues of Δ counted with multiplicity (equal to the number of linearly independent eigenfunctions with given eigen values).
Two compact Riemannian manifolds are said to be isospectral if they have the same spectrum.
The earlier question is more precisely: Does the spectrum determine the manifold upto isometry? Does it determine the shape of the manifold?

The prototypical examples of spaces we consider are quotients of the upper half plane by arithmetically defined discrete subgroup of $SL(2, R)$. For example, consider the group $SL(2, Z)$ consisting of integral 2 by 2 matrices with determinant one.
More generally one can define subgroups called $\Gamma_0(N)$ of $SL(2, Z)$ such that the entries of the matrix satisfy some divisibility properties in terms of the integer N.
The prototypical arithmetical spaces we consider are: the space of orbits of $\Gamma_0(N)$ acting on the upper half plane.
To be more precise, in the context of my work, we consider compact analogues of such spaces.

By arithmetic, we are working over integers Z or rational numbers Q. What we consider are polynomial equations in several variables with coefficients rational numbers (or more generally algebraic numbers), and the set of solutions X of such equations.

Examples:

- The Pythogorean equation:

$$x^2 + y^2 = z^2.$$

- The Fermat equation:

$$x^n + y^n = 1.$$

- the equation for an elliptic curve:

$$y^2 = x^3 + ax + b.$$

More generally one considers systems of such polynomial equations and their solution sets are called algebraic varieties (defined over integers). Not allowed are equations of the form:

$$y = sin(x).$$

Associated to such algebraically defined spaces X, can attach zeta functions.
For any given prime number p, let $N(X, p)$ number of solutions counted modulo p: if X is given by a single equation $f(x_1, ..x_n)$, a solution is a n-tuple of integers $a_1, ..a_n)$ such that $f(a_1, ..., a_n)$ is divisible by p.
Two solutions are equivalent if their difference is co-ordinate wise divisible by p.
And $N(f, p)$ counts the number of inequivalent solutions.
An approximation for the zeta function $Z(f, s)$ is the following:

$$Z(f, s)^{-1} = (1 - N(f, 2)/2^s)(1 - N(f, 3)/3^s)(1 - N(f, 5)/5^s)...$$

where the product runs over all prime numbers.
Example: Consider the equation $x = 0$. For each prime p you get only one solution. The corresponding zeta function is the Riemann zeta function,

$$1 + 1/2^s + 1/3^s + ...$$

Fundamenatal conjecture: The zeta functions of algebraic varieties have nice analytic properties.
Wiles during the course of proving Fermat's Last Theorem proved this conjecture in the special case of elliptic curves.
Coming back:
the quotients $X_0(N)$ of the upper half plane by the discrete subgroups $\Gamma_0(N)$, and their analogues have canonical arithmetical models.
Although we have defined them in a completely geometric manner, what this says is that these spaces also arise as solutions of some systems of polynomial equations with coefficients given by rational numbers.
So we can also consider the arithmetic of such spaces.
It is expected in some fundamental sense that these spaces are universal for the class of algebraic varieties.
Again coming back to Wiles theorem: this can be reinterpreted as saying that given any elliptic curve (defined by a cubic polynomial equation in 2 variables), there is an integer N and an onto map from

$$X_0(N) \to E$$

.

The basic conjecture or heuristic I make is the following:
The spectrum and the arithmetic of the class of arithmetical spaces defined above mutually determine each other.
Connection between the continuous and the discrete phenomena.
What we can show is the following:

- Verify that the spectrum determines the arithmetic in known construction of pairs of isospectral spaces which are not necessarily isometric.
- The other way around: it is known that complex conjugation preserves the arithmetic. We show that it also preserves the spectrum.
  This gives raise to new examples of pairs of isospectral but non-isometric spaces.