Sharp Inequalities and related problems
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$2xy \leq x^2 + y^2, \text{ for } x, y \in \mathbb{R}$
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xy \leq \frac{x^p}{p} + \frac{y^q}{q} \text{ where } \frac{1}{p} + \frac{1}{q} = 1, x, y \in \mathbb{R}
Inequalities: Discrete

\[ 2xy \leq x^2 + y^2, \text{ for } x, y \in \mathbb{R} \]

\[ xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1, x, y \in \mathbb{R} \]

For \( x_i, y_i \in \mathbb{R}, \ i = 1, ..., n \)

\[ \sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}} \]
Poincaré Inequality

Let $-\infty < a < b < \infty$ and $u \in C^1([a, b]), u(a) = u(b) = 0$
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$$|u(x)|^2 = \left| \int_a^x u'(t) \, dt \right|^2 \leq (b - a) \int_a^b |u'|^2 \, dt$$
Let $-\infty < a < b < \infty$ and $u \in C^1([a, b])$, $u(a) = u(b) = 0$

$$|u(x)|^2 = |\int_a^x u'(t) \, dt|^2 \leq (b - a) \int_a^b |u'|^2 \, dt$$

$$\frac{1}{(b - a)^2} \int_a^b |u|^2 \, dt \leq \int_a^b |u'|^2 \, dt$$
There exists an optimal constant $\lambda_1 > 0$ such that

$$\lambda_1 \int_a^b |u|^2 dt \leq \int_a^b |u'|^2 dt$$

holds for all functions $u \in C^1([a, b]), u(a) = u(b) = 0.$
Poincaré Inequality

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, then there exists an optimal constant $\lambda_1(\Omega) > 0$ such that

$$
\lambda_1(\Omega) \int_{\Omega} |u(x)|^2 \, dx \leq \int_{\Omega} |\nabla u(x)|^2 \, dx
$$

holds for all $u \in C_c^1(\Omega)$. 
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, then there exists an optimal constant $\lambda_1(\Omega) > 0$ such that

$$\lambda_1(\Omega) \int_\Omega |u(x)|^2 \, dx \leq \int_\Omega |\nabla u(x)|^2 \, dx$$

holds for all $u \in C^1_c(\Omega)$.

$\lambda_1(\Omega)$ is the first eigen value of $-\Delta$
Can we have this inequality in $\mathbb{R}^n$?

\[ C \int_{\mathbb{R}^n} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \quad u \in C^1_c(\mathbb{R}^n) \]
Inequalities in $\mathbb{R}^n$

Can we have this inequality in $\mathbb{R}^n$?

$$C \int_{\mathbb{R}^n} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \quad u \in C^1_c(\mathbb{R}^n)$$

Answer: NO
Can we have this inequality in $\mathbb{R}^n$?

$$C \int_{\mathbb{R}^n} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \quad u \in C^1_c(\mathbb{R}^n)$$

Answer: NO

More generally can we have

$$C \left[ \int_{\mathbb{R}^n} |u(x)|^q \, dx \right]^\frac{p}{q} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx, \quad u \in C^1_c(\mathbb{R}^n)$$
Suppose we have such an inequality for some $p, q$ then

$$q = \frac{np}{n-p} := p^*$$
Sobolev Inequality: Let $1 \leq p < n$, there exists an optimal constant $S_{p,n} > 0$ such that

$$S_{p,n} \left( \int_{\mathbb{R}^n} |u(x)|^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx , \quad u \in C^1_c(\mathbb{R}^n)$$
Sobolev Inequality

**Sobolev Inequality**: Let $1 \leq p < n$, there exists an optimal constant $S_{p,n} > 0$ such that

$$S_{p,n} \left[ \int_{\mathbb{R}^n} |u(x)|^{p^*} \, dx \right]^\frac{p}{p^*} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx, \quad u \in C_c^1(\mathbb{R}^n)$$

**Morrey’s Inequality**: Let $p > n$, there exists an optimal constant $S_{p,n} > 0$ such that

$$\sup_{x \neq y} \frac{u(x) - u(y)}{|x - y|^{1 - \frac{n}{p}}} \leq S_{n,p} \left[ \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \right]^\frac{1}{p}$$
First consider the case $p = 1$. i.e we have the inequality

$$S_{1,n} \left[ \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \, dx \right]^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)| \, dx \quad , \quad u \in C^1_c(\mathbb{R}^n)$$
First consider the case $p = 1$. i.e we have the inequality

$$S_{1,n} \left[ \left( \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \right] \leq \int_{\mathbb{R}^n} |\nabla u(x)| \, dx, \quad u \in C^{1}_c(\mathbb{R}^n)$$

$$S_{1,n} = n^{1-\frac{1}{n}} \left[ \omega_{n-1} \right]^{\frac{1}{n}}$$

where $\omega_{n-1} = H^{n-1}(\mathbb{S}^{n-1})$, the surface measure of the boundary of unit ball in $\mathbb{R}^n$. 
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$S_{1,n}$ is not achieved
Isoperimetric Inequality

Isoperimetric problem: Determine the shape of the closed plane curve having a given length and enclosing the maximum area.
Isoperimetric Inequality

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with smooth boundary then

$$H^{n-1}(\partial \Omega) \geq n^{1-\frac{1}{n}} [\omega_{n-1}]^{\frac{1}{n}} |\text{Vol } \Omega|^{\frac{n}{n-1}}$$

Inequality holds iff $\Omega$ is a ball
Equivalence

**Federer-Fleming**: Sobolev inequality with $p = 1$ is equivalent to the Isoperimetric Inequality
**Conjecture**: Let \((M, g)\) be a Cartan-Hadamard manifold then the isoperimetric inequality:

\[
H^{n-1}(\partial \Omega) \geq n^{1-\frac{1}{n}} [\omega_{n-1}]^{\frac{1}{n}} |\text{Vol } \Omega|^{\frac{n}{n-1}}
\]

holds in \((M, g)\).

- \(n=2\), Weil, 1926
- \(n=3\), Kleiner, 1992
- \(n=4\), Croke, 1984
- \(n \geq 5\), Joel Spruck, et al...2019 (???)
The case $p = 2$

\[
S_{2, n} \left[ \int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} \, dx \right]^{\frac{n-2}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx , \quad u \in C^1_c(\mathbb{R}^n)
\]
The case $p = 2$

\[
S_{2,n} \left[ \int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} \, dx \right]^{\frac{n-2}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx , \quad u \in C^1_c(\mathbb{R}^n)
\]

Extremal functions exist
The case $p = 2$

$$S_{2,n} \left[ \int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} \, dx \right]^{\frac{n-2}{n}} \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx , \quad u \in C_c^1(\mathbb{R}^n)$$

Extremal functions exist

If $u$ is an Extremal then $u$ solves

$$-\Delta u = u^{\frac{n+2}{n-2}} , \quad u > 0 , \quad \int_{\mathbb{R}^n} |\nabla u|^2 < \infty.$$
Extremals

- Rotational Symmetry
Extremals

- Rotational Symmetry
- Conformal invariance

Let $K : \mathbb{R}^n \to \mathbb{R}^n$ be a conformal map and $u$ is a solution of the PDE, the

$$\tilde{u} := J(K) \frac{n-2}{2} u(K(x))$$

is also a solution
• All solutions are of the form

\[ U(x) = \left( \frac{\sqrt{N(N-2)} \epsilon}{\epsilon^2 + |x - x_0|^2} \right)^{\frac{N-2}{2}} \]

for some \( \epsilon > 0 \) and \( x_0 \in \mathbb{R}^N \).
Obata : Let $g$ be a metric on the unit sphere $S^n$ conformal to the standard metric on $S^n$, then $g$ is of constant scalar curvature 1 iff $g$ is of constant sectional curvature 1.
Let \((M, g)\) be a compact Riemannian manifold of dimension \(N\) and scalar curvature \(K_g\), can we find a metric \(\tilde{g}\) conformal to \(g\) such that \(\tilde{g}\) has constant scalar curvature? If \(\tilde{g} = u^{\frac{4}{N-2}} g\), then this is equivalent to solving the PDE

\[-\frac{4(N-1)}{N-2} \Delta_g u + K_g u = ku^\frac{N+2}{N-2}\]

where \(k\) is a constant.
Let $2 \leq k < N$, $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^h$. Denote a point $x \in \mathbb{R}^N$ by $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^h$, then for $t \in [0, 2)$ there exists an optimal constant $S = S_{t,N,k} > 0$ such that

$$S \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*(t)}}{|y|^t} \, dx \right)^{\frac{2}{2^*(t)}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} \, dx$$

holds for all $u \in D^{1,2}(\mathbb{R}^N)$ where $2^*(t) = \frac{2(N-t)}{N-2}$ and $0 \leq \lambda \leq \frac{(k-2)^2}{4}$. 

Hardy-Sobolev -Maz’ya Inequality.
Euler Lagrange Equation.

\[-\Delta u - \lambda \frac{u}{|y|^2} = \frac{u^{p(t)-1}}{|y|^t}, \quad u > 0, \quad u \in D^{1,2}(\mathbb{R}^N)\]

where \(x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^h = \mathbb{R}^N\)
Classification of Solution

Question: Uniqueness of Solution
Difficulty: Lack of rotational symmetry
• (G Mancini, Sandeep) The problem has Hyperbolic symmetry and showed that the Solution space is of $N - k + 1$ dimensional
Classification of solutions when $t = 1$.

**Theorem**

(Fabri, Mancini, S) Let $u_0$ be the function given by

$$u_0(x) = u_0(y, z) = c_{N,k} \left( (1 + |y|^2 + |z|^2 \right)^{-\frac{N-2}{2}}$$

where $c_{N,k} = \{(N - 2)(k - 1)\}^{N-2 \over 2}$. Then $u$ is a solution of

$$-\Delta u = \frac{u^{N-2}}{|y|} \text{ in } \mathbb{R}^N, u > 0, u \in D^{1,2}(\mathbb{R}^N)$$

if and only if $u(y, z) = \lambda^{N-2 \over 2} u_0(\lambda y, \lambda z + z_0)$ for some $\lambda > 0$ and $z_0 \in \mathbb{R}^{N-k}$.
Thank You