Stochastic Approximation Algorithms with Set-Valued Maps

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November 10, 2019
Outline

1. Optimization under noise
2. Stochastic approximation algorithms
3. Random Directions Stochastic Approximation (RDSA)
4. Gradient algorithms with set-valued maps and their analysis
5. Ongoing and future work
Consider a repeated experiment that gives i.i.d input-output pairs \((X_n, Y_n), n \geq 0\), in real time.

**Goal:** Find a best parameterized fit

\[ Y_n = f_w(X_n) + \epsilon_n, \]

i.e., one with the least \(g(w) = \frac{1}{2}E[\| \epsilon_n \|^2]\)

\(f_w\) could correspond to polynomials, neural networks, splines, wavelets etc.

Note \(\nabla g(w) = -E[\langle Y_n - f_w(X_n), \nabla f_w(X_n) \rangle]\)

\(^1\)V.S.Borkar, Stochastic Approximation: A Dynamical Systems Viewpoint, Cambridge Univ Press, 2008
**Problem:** Cannot find the expectation

**Solution:** Drop the expectation!

Gradient Scheme with Noise:

\[ w_{n+1} = w_n + a(n) < Y_n - f_{w_n}(X_n), \nabla f_{w_n}(X_n) > \]

\[ = w_n + a(n)(-\nabla g(w_n) + M_{n+1}), \]

where \( M_{n+1} \) is the noise term

**Algorithms of this type are called Stochastic Approximation Algorithms**
Objective: Solve the equation $J(x) = 0$ when analytical form of $J$ is not known, however, ‘noisy’ measurements $J(x) + M_{n+1}$ can be obtained

\begin{equation}
x_{n+1} = x_n + a(n)(J(x_n) + M_{n+1})
\end{equation}

The Robbins-Monro Algorithm:

A Convergence Result

(C1) \( J : \mathcal{R}^N \to \mathcal{R}^N \) is Lipschitz continuous

(C2) \( \sum_n a(n) = \infty, \sum_n a(n)^2 < \infty \)

(C3) \( M_{n+1}, n \geq 0 \) is a martingale difference w.r.t. \( \{\mathcal{F}_n \triangleq \sigma(x_m, M_m, m \leq n)\} \). Further, for some \( K > 0 \),

\[
E[\| M_{n+1} \|^2 | \mathcal{F}_n] \leq K(1 + \| x_n \|^2)
\]

(C4) \( \sup_n \| x_n \| < \infty \) almost surely

- Let \( x^* \) be the unique globally asymptotically stable attractor for the ODE \( \dot{x}(t) = J(x(t)) \). Then

- **Theorem** (Convergence of SAA): Under (C1)–(C4), \( \{x_n\} \) converges almost surely to \( x^* \).

Let $J : \mathcal{R}^N \to \mathcal{R}$ be a given objective function having the form $J(x) = E_\mu[h(x, \mu)]$, where $\mu$ denotes ‘noise’ and $E_\mu[\cdot]$ is the expectation under that noise.

**Goal:** Find $x^* \text{ s.t. } J(x^*) = \min_{x \in \mathcal{R}^N} J(x)$.
Run two simulations with parameters

\[ x + \delta d = \begin{pmatrix} x^1 + \delta d^1 \\ x^2 + \delta d^2 \\ \vdots \\ x^N + \delta d^N \end{pmatrix}, \quad x - \delta d = \begin{pmatrix} x^1 - \delta d^1 \\ x^2 - \delta d^2 \\ \vdots \\ x^N - \delta d^N \end{pmatrix} \]

where \( d^1, \ldots, d^N \) are independent random variables with distribution \( U[-\eta, \eta] \)

Gradient Estimator:

\[ \hat{\nabla} J(x) = \frac{3}{\eta^2} \frac{d}{2\delta} \frac{J(x + \delta d) - J(x - \delta d)}{2\delta} \]

Hessian Estimator for RDSA

- Hessian Estimator:

\[
\hat{\nabla}^2 J(x) = \frac{9}{2\eta^4} R \left( \frac{J(x + \delta d) + J(x - \delta d) - 2J(x)}{\delta^2} \right),
\]

where

\[
R = \begin{bmatrix}
\frac{5}{2}(d^1)^2 - \eta^2 / 3 & \ldots & d^1 d^N \\
\ldots & \ddots & \ldots \\
\ldots & \ldots & \frac{5}{2}(d^N)^2 - \eta^2 / 3
\end{bmatrix}.
\]
Main Convergence Result

- **RDSA Algorithm:**
  \[ x_{n+1} = x_n - a(n)\Gamma(\hat{\nabla}^2 J(x_n))^{-1}\hat{\nabla} J(x_n) \]
  except that \( \delta \) is replaced with \( \delta_n \downarrow 0 \) and
  \[ \sum_n a(n) = \infty; \quad \sum_n \left( \frac{a(n)}{\delta_n} \right)^2 < \infty \] (2)

- Let \( x^* \) be the unique globally asymptotically stable equilibrium of the ODE
  \[ \dot{x}(t) = -\Gamma(\nabla^2 J(x))^{-1}\nabla J(x) \]

- Let \( a(n) = 1/n^\alpha \) and \( \delta_n = 1/n^\gamma \) with \( \alpha - \gamma > 0.5 \) and \( \beta \triangleq \alpha - 2\gamma > 0 \)

- **Theorem:** Under (C1) on \( \nabla J \), (2), (C3) and (C4)
  1. \( x_n \xrightarrow{a.s.} x^* \)
  2. \( n^{\beta/2}(x_n - x^*) \xrightarrow{dist} \mathcal{N}(\mu, \Omega) \)
Sufficient Conditions for Stability of $\text{SA}^5$

(C5)

(i) $J_c(x) \overset{\triangle}{=} J(cx)/c, \ c \geq 1$ satisfies $J_c \rightarrow J_\infty$, for some $J_\infty : \mathcal{R}^N \rightarrow \mathcal{R}^N$ uniformly on compacts

(ii) The origin in $\mathcal{R}^N$ is a unique globally asymptotically stable equilibrium for the ODE $\dot{x}(t) = J_\infty(x(t))$

(iii) There is a unique globally asymptotically stable equilibrium $x^* \in \mathcal{R}^N$ for the ODE $\dot{x}(t) = J(x(t))$

The Stability Theorem: Under (C1)-(C3), (C5), for any initial condition $x_0 \in \mathcal{R}^N$, $\sup_n \| x_n \| < \infty$ a.s. Further, $x_n \overset{a.s.}{\rightarrow} x^*$.

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Let $\hat{\nabla} J(x)$ denote an estimator for $\nabla J(x)$ s.t.

$$\| \hat{\nabla} J(x) - \nabla J(x) \| \leq \epsilon(\delta) \to 0 \text{ as } \delta \to 0$$

Consider the recursion $x_{n+1} = x_n - a(n)(\nabla J(x_n) + \epsilon_n)$, where $\| \epsilon_n \| \leq \epsilon \ \forall n$

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A set-valued map $h$ is called Marchaud if

- $h(x)$ is convex and compact for each $x$
- $\sup_{w \in h(x)} \| w \| \leq K(1 + \| x \|)$ for each $x$
- $h$ is upper-semicontinuous, i.e., given $\{x_n\} \subset \mathbb{R}^n$ and $\{y_n\} \subset \mathbb{R}^m$ with $x_n \to x$ and $y_n \to y$ with $y_n \in h(x_n)$, $\forall n$, we have $y \in h(x)$
Consider the differential inclusion (DI) in $\mathbb{R}^d$:

$$\dot{x}(t) \in H(x(t)), \quad (3)$$

where $H : \mathbb{R}^d \to \{\text{subsets of } \mathbb{R}^d\}$ is Marchaud. Then the above DI has at least one solution $x$ and each solution is absolutely continuous.\(^8\)

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The Set-Valued Semiflow $\Phi$ associated with (3) is defined on $[0, \infty) \times \mathcal{R}^d$ as

$$\Phi(t, x) = \{x(t) \mid x \in \Sigma, x(0) = x\},$$

where $\Sigma$ is the set of all absolutely continuous solutions to (3).

For $B \times L \subset [0, \infty) \times \mathcal{R}^d$, let

$$\Phi(B, L) = \bigcup_{t \in B, x \in L} \Phi(t, x)$$
$M \subset \mathbb{R}^d$ is invariant if for every $x \in M$, there exists $x \in \Sigma$ s.t. $x(t) \in M \ \forall t$ with $x(0) = x$
Attractor of a DI

- $A \subset \mathcal{R}^d$ is attracting if it is compact and there exists a neighborhood $U$ such that for any $\epsilon > 0$, $\exists T(\epsilon) \geq 0$ with
  $\Phi([T(\epsilon), \infty), U) \subset N^\epsilon(A)$

If the above $A$ is invariant, it is called an attractor
An Alternative View of Algorithm

- Recall the recursion

\[ x_{n+1} = x_n - a(n)(\nabla J(x_n) + \epsilon_n), \]

where \( \| \epsilon_n \| \leq \epsilon \ \forall n \)

- Alternatively consider

\[ x_{n+1} = x_n - a(n)g(x_n), \quad (4) \]

where \( g(x_n) \in G(x_n) \ \forall n \) and \( G(x) \triangleq \nabla J(x) + \bar{B}_\epsilon(0) \), i.e., gradient estimate lies in an \( \epsilon \)-ball around true gradient
Assumptions

- **(A1)** $\nabla J$ is a continuous function s.t. $\| \nabla J(x) \| \leq K(1 + \| x \|)$ for all $x \in \mathbb{R}^d$, $K > 0$

- **(A2)** $a(n) > 0 \forall n$ with

\[
\sum_n a(n) = \infty, \quad \sum_n a(n)^2 < \infty
\]

- Can show that $G$ is upper-semicontinuous

- Let $G_c(x) \triangleq \{ \frac{y}{c} \mid y \in G(cx) \}$

- Let $G_\infty(x) \triangleq \bar{co}(\text{Limsup}_{c \to \infty} G_c(x))$, where $\text{Limsup}_{c \to \infty} G_c(x) \triangleq \{ y \mid \liminf_{c \to \infty} d(y, G_c(x)) = 0 \}$
Lemma 1 The map $x \mapsto G_{\infty}(x)$ is Marchaud

Thus, $\dot{x}(t) \in -G_{\infty}(x(t))$ has at least one solution and which is absolutely continuous

(A3) $\dot{x}(t) \in -G_{\infty}(x(t))$ has an attractor set $A$ such that $A \subset B_{a}(0)$ for some $a > 0$ and $B_{a}(0)$ is a fundamental neighborhood of $A$

(A4) Let $c_{n} \geq 1$ be an increasing sequence of integers such that $c_{n} \uparrow \infty$ as $n \uparrow \infty$. Let $x_{n} \to x$ and $y_{n} \to y$ as $n \uparrow \infty$, such that $y_{n} \in G_{c_{n}}(x_{n}), \forall n$, then $y \in G_{\infty}(x)$
The Stability Result

- **Theorem 1** Under (A1)-(A4), the iterates (4) are stable i.e.,
  $\sup_n \| x_n \| < \infty$ a.s.

- Now recall that $G(x) = \nabla J(x) + \bar{B}_\epsilon(0)$

- Let the minimum set $M$ of $J$ be the global attractor of
  $\dot{x}(t) = -\nabla J(x(t))$

- It can be shown that any compact set $K$ with $M \subset K \subset \mathbb{R}^d$ is a
  fundamental neighborhood of $M$

- From Theorem 1, $\bar{x}(t) \in K_0 \ \forall t \geq 0$ for some (possibly sample path
  dependent) compact set $K_0$ which then is a fundamental neighborhood of $M$
The Main Result

**Theorem 2** Given $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that (4) converges to $N(\delta)(M)$ provided $\epsilon \leq \epsilon(\delta)/2$
Related Recent Work

- A general stochastic recursion with set-valued maps and Markov noise
  
  $$x_{n+1} = x_n + a(n)(h(x_n, Z_n) + M_{n+1})$$

- General convergence with $Z_n$ non-ergodic, iterate-dependent, Markov process [V.Yaji and SB, *Stochastics*, 2018]

- Two-timescale stochastic recursions with Markov noise
  
  $$x_{n+1} = x_n + a(n)(h(x_n, y_n, Z^1_n) + M^1_{n+1})$$
  $$y_{n+1} = y_n + b(n)(g(x_n, y_n, Z^2_n) + M^2_{n+1})$$

- $Z^1_n, Z^2_n$ independent non-ergodic iterate-dependent Markov processes, $M^1_{n+1}, M^2_{n+1}$ independent martingale differences, $a(n) = o(b(n))$, $h, g$ point-to-point maps – analysis and application to reinforcement learning [P.Karmakar and SB, *Math of OR*, 2018]

- Analysis under set-valued $h, g$ [V.Yaji and SB, *Math of OR*, 2019]
Ongoing and Future Work

- Finding minima of non-differentiable functions under noise
- Algorithms for convergence to global minima
- Asynchronous update algorithms
- Reinforcement learning algorithms for partially observed Markov decision processes
- Analysis of deep reinforcement learning algorithms
- Applications in robotics, microgrids, vehicular traffic control etc.