Kinetic properties of Alfvén waves

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1. Introduction

The Alfvén wave is probably one of the first waves found to exist in a plasma. However, until recently relatively less attention has been paid to it, with an exception in astrophysical cases, due to difficulties in exciting it in laboratory scale plasmas. As the scale size of the laboratory plasma is increased along with the progress in thermonuclear fusion research, its importance is inevitably becoming clear. In this lecture, we present linear and nonlinear properties of the Alfvén wave in a collisionless, magnetized plasma. The effect of finite perpendicular (with respect to the magnetic field) wavelength is introduced whereby the Alfvén wave accompanies the electric field in the direction parallel to the magnetic field. This leads to various wave-particle interactions which are absent in the classic magnetohydrodynamic treatment.

The talk will be divided into three sections. In section 1, linear properties of the kinetic Alfvén wave will be discussed. They include the linear dispersion relation and the excitation mechanisms of the kinetic Alfvén wave by drift wave instability or the resonant mode conversion. In particular the resonant absorption of a magnetohydrodynamic Alfvén wave in a nonuniform plasma is shown to be a manifestation of the resonant mode conversion of the magnetohydrodynamic mode into the kinetic Alfvén wave.

The nonlinear properties of the kinetic Alfvén wave will be discussed in section 2. Here the quasilinear theory will first be presented to show that the kinetic Alfvén wave turbulence can cause various anomalous transport phenomena. We will discuss anomalous diffusion, anomalous viscosity and resonant heating. The nonlinear wave-wave and wave-particle interactions will be discussed next. Here the interactions of two kinetic Alfvén waves with an ion acoustic wave (stimulated Brillouin scattering) or with ions (stimulated Compton scattering) are considered. It is shown that the nonlinear coupling coefficient is larger than that of a magnetohydrodynamic Alfvén wave by a factor of \( \omega_{ci}/\omega \), where \( \omega_{ci} \) and \( \omega \) are the ion cyclotron frequency and the Alfvén wave frequency. Finally in this section, the kinetic Alfvén wave is shown to have an exact solitary wave solution that propagates in a direction oblique to the ambient magnetic field.

Section 3 is devoted to the application of the kinetic Alfvén wave to heat a Tokamak-type plasma to the thermonuclear temperature. In spite of the drawback that the coupling coil should be installed inside the plasma chamber, this heating method is shown to be superior to other RF heating schemes as well as to the neutral beam...
heating. Only the RF field (∼1 MHz) of a magnetic field amplitude of four Gauss is shown to be needed to provide 200 MW RF power to a reactor size plasma.

2. Linear theory

2.1. Dispersion relation

Here we derive the dispersion relation of the kinetic Alfvén wave. We consider an electromagnetic perturbation at a frequency much below the ion cyclotron frequency \( \omega_{ci} \). From the well known theory of the magnetohydrodynamic waves, we expect to have three basic waves in this frequency range, the Alfvén wave and the slow and fast magnetosonic waves. If we consider a perturbation with a wavelength, in the direction perpendicular to the magnetic field, \( 2\pi/k_z \), comparable to the ion gyroradius \( \rho_i \), only the Alfvén and slow magnetosonic (ion acoustic wave) can be excited in a low beta plasma because the magnetosonic mode, which is the isotropic mode, has a frequency comparable to or larger than the ion cyclotron frequency for such a short wavelength.

This fact allows us to assume that such a perturbation accompanies little change in the flux density of the magnetic field. We take a coordinate system such that the \( z \) axis corresponds to the direction of the ambient magnetic field, this means that the \( z \) component of the magnetic field perturbation \( B_z \) is almost zero. In such a case, we can use the two potentials \( \varphi \) and \( \psi \) to represent the entire field variables [Kadomtsev, 1965]. \( \varphi \) and \( \psi \) are defined as

\[
E_\perp = - \nabla_\perp \varphi, \\
E_\parallel = E_z = - \nabla_\parallel \psi,
\]

where \( \perp \) and \( \parallel \) indicate components perpendicular and parallel to the ambient magnetic field. For an electrostatic perturbation \( \varphi = \psi \), but for an electromagnetic perturbation \( \varphi \neq \psi \).

For an illustrative purpose, we first derive the dispersion relation for a plasma with cold ions and hot electrons. We will see that the essential feature of the kinetic Alfvén wave is retained even in this ideal case. For the Maxwell’s equation, we use quasi-neutrality condition,

\[
n_i = n_e,
\]

for the perturbed ion and electron densities \( n_i \) and \( n_e \), and the Ampère’s law in the parallel direction, which can be written, using \( \varphi \) and \( \psi \), as

\[
\frac{\partial}{\partial z} \nabla_\perp^2 (\varphi - \psi) = \mu_0 \frac{\partial}{\partial t} (J_{iz} + J_{iz}),
\]

where \( J_i \) and \( J_e \) are the current densities of ions and electrons and \( \mu_0 \) is the permeability of space. For cold ions, the ion density \( n_i \) can be derived from the equation of continuity,

\[
\frac{\partial n_i}{\partial t} + \nabla_\perp (v_{i0} n_i) + \nabla_\parallel (v_{iz} n_0) = 0,
\]
where, $v_{td}$ is the ion perpendicular drift velocity given by

$$v_{td} = \frac{E \times B_0}{B_0^2} + \frac{m_i}{eB_0^2} \frac{dE_1}{dt}. \quad (6)$$

Here $m_i$ is the ion mass, $e$ is the absolute value of the electron charge and $B_0$ is the flux density of the ambient magnetic field. $v_{iz}$ is obtained from the equation of motion in the $z$ direction,

$$\frac{dv_{iz}}{dt} = -\frac{e}{m_i} \frac{\partial \psi}{\partial z}. \quad (7)$$

If we linearize eqs (5) to (7) and use Fourier amplitude expression $\exp i(k_\perp \cdot x_\perp + k_z z - \omega t)$, we obtain the ion number density as

$$n_i = \frac{\varepsilon_0}{\varepsilon} \left[ \frac{-\omega_{pi}^2 k_z^2 \varphi + \omega_{pi}^2 \omega^2 k_z^2 \psi}{\omega_{ci}^2} \right], \quad (8)$$

where $\omega_{pi}$ is the ion plasma frequency and $\varepsilon_0$ is the space dielectric constant. The electron density should obey the Boltzmann distribution

$$n_e = \frac{\varepsilon_0}{\varepsilon} \frac{n_0}{v_{Te}^2} \psi, \quad (9)$$

where $\omega_{pe}$ is the electron plasma frequency, $v_{Te}$ is the electron thermal speed, and $T_e$ is the electron temperature.

The current density in the $z$ direction $J_z$ can be obtained from the number density as

$$\mu_0 J_z = \frac{\omega_{pi}^2 k_z \psi - \omega_{pe}^2 \omega \psi}{c^2} \quad (10)$$

If we use these results in the Maxwell’s eqs (3) and (4), we obtain the following dispersion relation,

$$\left(1 - \frac{\omega^2}{k_z^2 c_s^2} \right) \left(1 - \frac{\omega^2}{k_z^2 v_A^2} \right) = \frac{\omega^2 k_z^2}{k_z^2 \omega_{ci}^2}, \quad (11)$$

where $c_s = (T_e/m_i)^{1/2}$ is the ion acoustic speed, and $v_A = (B_0^2/m_i n_0 \omega_0)^{1/2}$ is the Alfvén speed. As expected we obtained the dispersion relation which shows coupling between the ion acoustic wave and the Alfvén wave. These modes decouple for a low beta plasma. The dispersion relation of the kinetic Alfvén wave then becomes,

$$\omega^2 = k_z^2 v_A^2 \left(1 + k_\perp^2 \rho_s^2 \right), \quad (12)$$

where $\rho_s = c_s/\omega_{ci}$ is the equivalent ion gyroradius. Compared with the magnetohydrodynamic Alfvén wave, we see that the dispersion relation of the kinetic Alfvén wave depends on $k_\perp$, and the wave can propagate in the direction perpendicular to the ambient magnetic field.
To include the effect of finite ion temperature and Landau damping by both species, we must use Vlasov equation. The dispersion relation then is shown to be modified to (for $\beta \ll 1$) [Hasegawa and Chen 1976],

$$\frac{\omega^2}{k_z^2 v_A^2} = \frac{\lambda_i}{1 - I_0(\lambda_i) e^{-\lambda_i}} + \lambda_e \left( 1 - i\delta_e - i \frac{v_{Ti}^2}{v_A^2} \delta_i \right),$$

where $\lambda_i = k_z^2 v_{Ti}^2 (\omega^2 c_i^2)$, $\lambda_e = k_z^2 v_{Te}^2 = \lambda_i(T_e/T_i)$, $I_0(\lambda)$ is the modified Bessel function of the first kind and $\delta_i$ and $\delta_e$ are respectively ion and electron Landau damping rate given by

$$\delta_i = 2 \sqrt{\pi} \beta_i^{-3/2} \exp(-\beta_i^{-1}),$$

$$\delta_e = \sqrt{\pi} \beta_i^{-3/2} (T_e/T_i)^{1/2} \left( m_e/m_i \right)^{1/2}.$$

Here $\beta_i (=2v_{Ti}^2/v_A^2)$ is the ratio of ion pressure to the magnetic field pressure and $v_{Ti}$ is the ion thermal speed.

We can see here that the kinetic Alfvén wave faces Landau damping due to the presence of the parallel electric field $E_z$.

The phase and amplitude relation among different field components are given, for example, using the potential $\varphi$,

$$E_x = - i k_x \varphi,$$

$$E_y = - i k_y \varphi,$$

$$E_z = i k_z (T_e/T_i) \left[ 1 - I_0 \exp(-\lambda_i) \right] \varphi,$$

$$B_x = i (k_y k_z / \omega) \left[ 1 + (T_e/T_i) \left[ 1 - I_0 \exp(-\lambda_i) \right] \right] \varphi,$$

$$B_y = - i (k_x k_z / \omega) \left[ 1 + (T_e/T_i) \left[ 1 - I_0 \exp(-\lambda_i) \right] \right] \varphi.$$  

2.2. Excitation of kinetic Alfvén wave by drift wave instability

Let us now discuss the way by which the kinetic Alfvén wave can be excited. The most well known process is the excitation by the drift wave instability. As was shown by Mikhailovskii (1967), the drift wave instability, in a finite beta plasma, excites the Alfvén wave. Since the drift wave has a perpendicular wavelength comparable to the ion gyroradius, the excited Alfvén wave is actually the kinetic Alfvén wave. The dispersion relation is given by (Hasegawa 1975a)

$$\frac{\omega^2}{v_A^2} = \frac{T_e}{T_i} \lambda_i k_x^2 v_A^2 \left( \omega - \omega_i^* \right) \left[ 1 + i \sqrt{\frac{\pi}{2}} \frac{\omega}{k_x v_T e} \right],$$

$$= \frac{T_e}{T_i} \lambda_i k_x^2 v_A^2 \left( \omega - \omega_i^* \right),$$

(17)
where \( \omega_i^* \) and \( \omega_e^* \) are drift wave frequencies of electrons and ions given respectively by

\[
\omega_i^* = \frac{k_y \kappa v_T e^2}{\omega_{ci}}, \tag{18}
\]

\[
\omega_e^* = - \frac{k_y \kappa v_T i^2}{\omega_{ce}}, \tag{19}
\]

and \( \kappa \) is the measure of the density gradient in \( x \) direction, namely

\[
\kappa = \frac{\partial \ln n_0(x)}{\partial(x)}. \tag{20}
\]

### 2.3. Excitation of kinetic Alfvén wave by resonant mode conversion

Another way by which the kinetic Alfvén wave is excited in the plasma is the resonant mode conversion. This is a less well known process than the drift wave instability but probably more likely to occur.

Since the magnetohydrodynamic Alfvén wave has an anisotropic dispersion relation given by \( \omega = k \parallel v_A \), there exists no discrete eigenmode in a nonuniform plasma in which the plasma density (and/or magnetic field intensity) varies in the perpendicular direction (Uberoi 1972). However, when the plasma density variation is discrete, (sharp variation) a magnetohydrodynamic surface wave becomes the eigenmode of the system. For example, when the plasma density varies from zero to \( n_0 \) in a step form, the surface eigen frequency \( \omega_s \) is given by (Chen and Hasegawa 1974)

\[
\omega_s = \sqrt{2} k \parallel v_A, \tag{21}
\]

where \( v_A \) is the Alfvén speed corresponding to the plasma density \( n_0 \).

Such a surface Alfvén wave may be excited by an external impulse (Chen and Hasegawa 1974) or by the classic kelvin-Helmholtz instability (Chandrasekhar 1961).

Now, if the density is allowed to vary smoothly from zero to \( n_0 \) (in the \( x \) direction), the surface eigen frequency \( \omega_s \) in general resonates with a local Alfvén frequency \( k \parallel v_A(x) \) at a local point \( x_0 \) where \( \omega_s = k \parallel v_A(x_0) \) is satisfied. When this local resonance (called spatial resonance) occurs, the surface wave is known to be absorbed (Uberoi 1972; Chen and Hasegawa 1974; Ohsawa et al 1976).

In fact, if the ideal magnetohydrodynamic equation is used, the wave equation for the \( y \) component of the electric field near the resonant point can be written as,

\[
\frac{d}{dx} \left\{ B_y^2 \left( \frac{\omega^2}{v_A^2(x)} - k^2_x \right) \frac{dE_y}{dx} \right\} = 0. \tag{22}
\]

Hence, near the point \( x=x_0 \) at which \( \omega^2/v_A^2(x_0) - k^2_x = 0 \), the solution has a logarithmic singularity and is given by

\[
E_y = c \ln(x-x_0) \quad x>x_0 \tag{23}
\]

\[
= c \left\{ \ln |x-x_0| + i\pi \right\} \quad x>x_0.
\]

The imaginary part of the solution produces the collisionless absorption discussed above.
The singularity appears in consequence of the idealized magnetohydrodynamic approximation; the solution within a distance finer than the ion gyroradius does not have any meaning.

It was shown by Hasegawa and Chen (1975) that if the kinetic theory is used, the singularity disappears, and instead, the wave equation becomes fourth order and represents the resonant mode conversion from the surface Alfvén wave to the kinetic Alfvén wave. The resonant absorption is a manifestation of a dissipation of the Alfvén surface wave due to the resonant mode conversion to the kinetic Alfvén wave which convects the surface wave energy away from the resonant layer.

If a linear density profile \( n_0(x) = \kappa x \) is assumed, the solution of the fourth order wave equation near the resonant point \( (x=0) \) can be written (Hasegawa and Chen 1976) in terms of the \( x \) component of the wave electric field, (cf figure 1),

\[
E_x = \begin{cases} 
-\frac{\sqrt{\pi} E_0}{(k\rho)^{2/3}} \left( \frac{\Delta}{x^3} \right)^{1/4} \exp \left\{ i \left[ \frac{2}{3} \left( \frac{x}{\Delta} \right)^{3/2} + \frac{\pi}{4} \right] \right\} + \frac{E_0}{k\kappa x} & x > 0, \end{cases}
\]

\[
= \frac{E_0}{k\kappa x} & x < 0,
\]

where \( \Delta = (\rho^2 / \kappa)^{1/3} \) and

\[
\rho^2 = \left( \frac{3}{4} + \frac{k^2 v_A^2 T_e}{\omega^2 T_i} \right) \rho_i^2,
\]

![Figure 1. Schematic diagram of the \( x \) component (radial component) of the wave electric field near the spatial resonant point, \( x=0 \). \( x>0 \) region corresponds to the higher density side where the kinetic Alfvén wave is excited. The plasma heating occurs when the kinetic Alfvén wave is dissipated by wave particle interactions. The wavelength \( d \) near the resonant point is approximately given by \((3/18)^{1/3} (\rho^2 / \kappa)^{1/3} \sim 50 \rho_i\).](image)

evaluated at $x=0$. $E_0$ is an integration constant representing a nominal value of $E_z$ at a large negative $x$, and corresponds to the amplitude of the magnetohydrodynamic mode.

As is expected, the kinetic Alfvén wave propagates in the higher density side ($x>0$) of the resonant point as can be seen in eq. (24). Also in these expressions, it is worth noting that the first few peak amplitudes of the kinetic Alfvén wave after the mode conversion are given by $E_0 (k_p)^{-2/3}$ with effective wave number of $(\kappa/\rho^2)^{1/3}$, while away from the resonant point, $x \sim \kappa^{-1}$, the amplitude and the wave number of the kinetic Alfvén wave become $E_0(k_p)^{-1/2}$ and $\rho^{-1}$, respectively.

3. Nonlinear Theory

3.1. Quasilinear effects and anomalous transport coefficients

In this section, we show that the kinetic Alfvén wave can cause anomalous transport through the Čerenkov resonance, $\omega=k_\parallel v_\parallel$. We derive anomalous diffusion and anomalous viscosity coefficients as well as resonant heating rate produced by the kinetic Alfvén wave turbulence.

3.1.a Anomalous cross B diffusion

In this section, we discuss the plasma diffusion across the ambient magnetic field caused by the kinetic Alfvén turbulence. We use the quasi-linear theory which is commonly used in deriving a turbulent transport coefficient (Sagdeev and Galeev 1969).

We first consider electron dynamics. The suitable drift kinetic equation for the electron drift distribution function $f_e(v_\parallel, x, t)$ is (Hasegawa 1975).

$$\frac{\partial f_e}{\partial t} + v_z \frac{\partial f_e}{\partial z} + \nabla_\perp \cdot (v_d f_e) - \frac{e}{m_e} \left[ E_z + (v_d \times B_\perp) \cdot e_z \right] \frac{\partial f_e}{\partial v_z} = 0. \quad (26)$$

Here we have used a local cartesian coordinate system in which the $z$ axis is taken in the direction of the ambient magnetic field $B_0$; $E$ and $B$ are the wave electric and magnetic fields with subscripts $z$ and $\perp$ indicating the components in the direction parallel and perpendicular to the ambient magnetic field respectively. $v_d$ is the drift velocity perpendicular to the magnetic field which, for electrons, is given by

$$v_d = \frac{E_\perp \times B_0}{B_0^2} + \frac{B_\perp}{B_0} v_z. \quad (27)$$

We have also assumed that the plasma $\beta$ is small (compared with unity) to warrant neglecting of the magnetic compression $B_z$.

We assume that the density gradient exists in the $x$ direction. Then the quasi-linear equation for the average distribution function $\langle f_e \rangle$ can be written as

$$\frac{\partial \langle f_e \rangle}{\partial t} + \frac{1}{B_0} \frac{\partial}{\partial x} \left( E_y + v_z B_z \right) \langle f_e \rangle - \frac{e}{m_e} \langle E_z \frac{\partial f_e}{\partial v_z} \rangle = 0, \quad (28)$$
where $f_e^{(1)}$ is the linear response and $\langle \rangle$ means the ensemble average. If we use the Fourier amplitude expression for the linear terms, such as $f_e^{(1)} = \text{Re} \sum_k f_k^{(1)} (x)$
e^{i(k \cdot x - \omega t)}$, the linear response of the distribution function $f_k^{(1)}$ can be expressed as

$$f_k^{(1)} = iP \left[ \frac{1}{k_z v_z - \omega} \left( \frac{k_z v_z}{\omega B_0} \frac{\partial \langle f_e \rangle}{\partial x} - \frac{e}{m_e} \frac{\partial \langle f_e \rangle}{\partial v_z} \right) E_{z \mathbf{k}} \right.$$

$$\left. - \frac{1}{\omega B_0} \frac{\partial \langle f_e \rangle}{\partial x} E_{y \mathbf{k}} \right]$$

$$+ \pi \delta (k_z v_z - \omega) \left( \frac{e}{m_e} \frac{\partial \langle f_e \rangle}{\partial v_z} - \frac{k_z v_z}{\omega B_0} \frac{\partial \langle f_e \rangle}{\partial x} \right) E_{z \mathbf{k}},$$

where $P$ indicates the principal value, $\delta$ the Dirac's $\delta$ function, and the Maxwell's equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ is used to express $B_z$ in terms of $E_y$ and $E_z$;

$$B_{z \mathbf{k}} = (k_y E_{z \mathbf{k}} - k_x E_{y \mathbf{k}})/\omega.$$  

(30)

If we substitute eq. (29) into (28), the term multiplied by the delta function, which represents the dissipation due to the Landau damping, contributes to the quasi-linear diffusion coefficient. The first term contributes to the diffusion in velocity space, and the second term to that in coordinate space. Taking the second term contribution here, we have

$$\frac{\partial \langle f_e \rangle}{\partial t} = \frac{\partial}{\partial x} \sum_k \left[ \frac{k_y^2 v_z^2}{2 \omega B_0^2} |E_{z \mathbf{k}}|^2 \pi \delta (k_z v_z - \omega) \frac{\partial \langle f_e \rangle}{\partial x} \right].$$

(31)

As is expected only the $z$ component of the wave electric field contributes to the diffusion. The above result is valid for any kind of low frequency electromagnetic waves and is not restricted to the kinetic Alfvén wave.

The diffusion equation for electron density $n_e = \int \langle f_e \rangle \, dv_z$ can be constructed by integrating eq. (31) over $v_z$. If we assume a Gaussian distribution for $\langle f_e \rangle$, i.e.,

$$\langle f_e \rangle = n_e (x) \frac{1}{\sqrt{2\pi} v_{Te}^0} e^{-v_z^2/2v_{Te}^0}$$

(32)

we have, for $v_{Te} \gg v_A$,

$$\frac{\partial n_e}{\partial t} = \frac{\partial}{\partial x} \left( D_e \frac{\partial n}{\partial x} \right),$$

(33)

where

$$D_e = \sqrt{\frac{\pi}{8}} \sum_k \frac{k_y^2 |E_{z \mathbf{k}}|^2}{k_z^2 B_0^2} \frac{1}{|k_z| v_{Te}}.$$  

(34)

As was stated previously, a unique feature of the kinetic Alfvén wave is that it accompanies the electric field $E_z$ in the direction of the ambient magnetic field.
We now discuss the ion diffusion. For simplicity we assume that $T_e > T_i$ and ignore the terms involving $\lambda_i$. In effect this is equivalent to the assumption of hot electrons and cold ions. However, the essential feature of the diffusion coefficient can be retained.

The average density of ions follows the conservation law,

$$\frac{\partial \langle n_i \rangle}{\partial t} + \frac{\partial}{\partial x} \langle n_i v_x \rangle = 0,$$

where the ion velocity is given by

$$v_x = \frac{E_y}{B_0},$$

to the lowest order in $\lambda_i$. The number density perturbation $n_{ik}$ should be the same as the electron density perturbation, $\int f_k^{(1)} dv_x$, from the quasineutrality requirement.

$$n_{ik} = n_0 \int f_k^{(1)} dv_x$$

$$= \frac{\epsilon n_0}{k_z T_e} \left( 1 + \frac{k_z T_e}{eB_0} \frac{1}{n_0} \frac{\partial n_0}{\partial x} \right) \left( 1 + i \delta e \right) \frac{E_{zik}}{\frac{E_y k}{\omega B_0}}.$$  (37)

If we substitute this expression into eq. (35), we see that correlation between $E_z$ and $E_x$ needs to be evaluated. To obtain the relation between $E_z$ and $E_x$, we use again the quasineutrality condition with the ion density perturbation being given by, from the equation of continuity;

$$n_{ik} = n_0 \left( \frac{1}{n_0} \frac{\partial n_0}{\partial x} \frac{E_{zik}}{B_0} - \frac{e k_z}{m_i \omega} E_{xik} + \frac{k_z \omega m_i}{e B_0^2} E_{yik} \right).$$  (38)

From eqs (37) and (38), we have

$$E_{yik} = \frac{k_y}{k_z} \frac{1}{\lambda_i} \left[ \frac{k_z c_s^2}{\omega^2} + \left( 1 + \frac{k_z T_e}{eB_0} \frac{1}{n_0} \frac{\partial n_0}{\partial x} \right) \left( 1 + i \delta e \right) \right] E_{zik}. $$  (39)

Substituting eqs (38) and (39) into eq. (35), we can obtain the ion diffusion coefficient $D_i$.

$$D_i = \sqrt{\frac{\pi}{8}} \sum_k \frac{\omega c_s^2}{\omega^2 B_0^2} \frac{1}{|k_z| v_{Te}} |E_{zik}|^2.$$  (40)

It is interesting to compare this result with the electron diffusion coefficient given in eq. (34). Using eq. (34) we see

$$D_i = \frac{1}{\lambda_i} \left( \frac{k_z c_s}{\omega} \right)^2 D_e \approx \frac{\beta}{2\lambda_i} D_e.$$

This means that for a low $\beta$ plasma, the anomalous diffusion of ions is smaller than that of electrons. This result is opposite to the classic diffusion. We also note that for an electrostatic turbulence where $\omega \sim k_z c_s$, eq. (41) gives $D_i \sim D_e/\lambda_i \sim D_e$. This is consistent with the well-known anomalous diffusion by electrostatic drift wave instability (Sagdeev and Galeev 1969).
3.1.b Anomalous viscosity

In this section, we derive the anomalous viscosity for a plasma flow parallel to the magnetic field.

The parallel viscosity may be obtained by using the drift kinetic equation for electrons and a shifted velocity distribution function in which the average velocity \( v_0 \) is made to depend on the coordinate \( x \). Thus

\[
\frac{\partial \langle f_e \rangle}{\partial x} = \frac{v_z - v_0 \frac{\partial v_0}{\partial x}}{\frac{T_e}{v_0}} \langle f_e \rangle. \tag{42}
\]

Substituting this expression into eq. (29) and taking average of eq. (28), we have

\[
\frac{\partial \langle f_e \rangle}{\partial t} = \frac{\partial}{\partial x} \sum_k k_z^2 v_z^2 \left| \frac{E_{zk}}{k^2} \right|^2 \frac{T_e}{v_0^2} \delta (k_z v_z - \omega) \frac{v_z - v_0}{\frac{T_e}{v_0}} \frac{\partial v_0}{\partial x} \langle f_e \rangle. \tag{43}
\]

If we multiply (43) by \( v_z \) and integrate it over \( v_z \), we have

\[
\frac{\partial v_0}{\partial t} = \frac{\partial}{\partial x} \left( \eta_\parallel \frac{\partial v_0}{\partial x} \right), \tag{44}
\]

where the parallel viscosity \( \eta_\parallel \) is given by

\[
\eta_\parallel = \sqrt{\frac{\pi}{8}} \sum_k \frac{1}{|k_z| v_T e} \left| \frac{E_{zk}}{B_0^2} \right|^2 \frac{\lambda_z^2}{v_T e^2} \left( 1 + \lambda_z \right) \frac{v_T e}{v_0^2} \frac{v_T e}{v_0^2}
\]

\[
= \sqrt{\frac{\pi}{8}} \sum_k \frac{1}{|k_z| v_T e} \left| \frac{B_{zk}}{B_0^2} \right|^2 \frac{\lambda_z^2}{v_T e^2} \frac{v_T e}{v_0^2}, \tag{45}
\]

and \( T_e T_i \) is assumed for simplicity. If we compare \( \eta_\parallel \) with the cross B diffusion coefficient, \( D_e \), \( \eta_\parallel \) is found to be smaller than \( D_e \) by a factor of \( (v_A/v_T e)^2 \). This difference originates from the fact that the viscosity coefficient comes from the higher moment of the distribution function.

3.1.c Resonant heating of electrons

We now consider resonant heating of electrons by the kinetic Alfvén wave turbulence. When the plasma is relatively cold (\( \sim \) a few eV), the resonant condition, \( v_T e = v_A \), may be satisfied. We consider here the heating of such cold electrons by the kinetic Alfvén wave. The heating rate \( n_0 dT_e/dt \) can be obtained as

\[
n_0 \frac{dT_e}{dt} = R_e \frac{1}{2} (J_{zk} E_{zk})
\]

\[
= n_0 T_e \omega \sqrt{\frac{\pi}{8}} \frac{m_t}{m_e} \sum_k \left| \frac{B_{zk}}{B_0^2} \right|^2 \frac{\lambda_z}{(1 + \lambda_z)^{3/2}} \times F \left( \frac{v_A}{v_T e} \right), \tag{46}
\]

where \( F \) is a function which represents the wave-particle resonance,

\[
F(x) = x^3 \exp \left( -x^2/2 \right). \tag{47}
\]
This is a sharply peaked function having the maximum value $\sim 1.2$ near $x = 1.7$. This indicates that if $v_A \sim 2v_T$, because of the factor of the large mass ratio in eq. (46), heating rate becomes large even if the wave amplitude $B_{x,0}$ is on the order of one percent of the ambient field $B_0$.

### 3.2. Decay of the kinetic Alfvén wave

In this section, we consider nonlinear wave-wave and wave-particle interactions of the kinetic Alfvén wave. As an example we study decay processes from a kinetic Alfvén wave into another kinetic Alfvén wave and an ion acoustic wave.

The ion acoustic wave can be either weakly or heavily damped depending on the temperature ratio $T_e/T_i$. In the former case $T_e > T_i$, resonant decay occurs. In the latter case $T_e \leq 5T_i$, the decay instability is made through nonlinear ion Landau damping; i.e., it is an induced scattering process. Both these two cases are considered here.

In order to illustrate the physics more clearly, we adopt a simple model of a uniform plasma to which a self-consistent pump wave $\varphi_0(t)$ (the kinetic Alfvén wave) of the following form

$$\varphi_0(x, t) = \frac{1}{2} \{ \varphi_0 \exp \left[ -i (\omega_0 t - k_0 \cdot x) \right] + c.c. \}, \quad (48)$$

is applied. Here, $(\omega_0, k_0)$ satisfies the linear dispersion relation for kinetic Alfvén wave. The pump field $\varphi_0$ is assumed to be sufficiently weak so that only interactions up to $O(\varphi_0^2)$ need to be kept. Furthermore, since decay instabilities are considered here, we ignore the upper sideband as being off-resonant and only discuss the couplings among the pump wave $(\pm \omega_0, \pm k_0)$, the lower-sideband $(\omega_-, k_-) = (\omega_0 - \omega_0, k_0 - k_0)$ and the low frequency wave $(\omega_0, k_0)$. Note, again $(\pm \omega_0, \pm k_0)$ and $(\omega_0, k_0)$ are kinetic Alfvén waves, and $(\omega_0, k_0)$ is the ion acoustic mode. For low-$\beta$ plasmas, $|\omega_0| > |k_x V_A| > |\omega_s| > |k_x c_s|$. To further simplify the analyses, we make the additional assumptions that $|k_x p_i| < 1$ and $T_e/T_i > 1$, so that we need only keep the effects of finite electron inertia. The dynamics of both species are then described by the following drift kinetic equation

$$\frac{\partial f_j}{\partial t} + v_z \frac{\partial f_j}{\partial z} + \nabla \cdot (v_j f_j) + \left( \frac{q}{m_j} \right) [E_z + (v \times B_j) \cdot e_z] \frac{\partial f_j}{\partial v_z} = 0, \quad (49)$$

where

$$v_j = (v_E + v_P + v_B), \quad j = e, i, \quad (50)$$

with $v_E = E_0 \times B_0 /[B_0^2]$, $v_P = (m/qB_0^2)(dE_0/|dt|)$ and $v_B = v_e B \parallel /B_0$. $f_j(x, v_z, t)$ is the drift distribution function and other notations are standard. Note here, $d/dt$ contains a convective term; i.e., $d/dt = \partial /\partial t + (v \cdot \nabla)$.

Let $f_j = f_j^{(0)} + f_j^{(1)} + f_j^{NL}$, then using the two potentials $\varphi$ and $\psi$ defined in eqs (1) and (2), we obtain the linear response $f_j^{(1)}$ as

$$f_{e, i}^{(1)} = f_{e, i}^{(1)} \Psi \quad (51)$$

and

$$f_i^{(1)} = -\lambda f_i^{(0)} \varphi + c_i \frac{\partial f_i^{(0)}}{\partial v_z} \frac{\partial \varphi}{\partial \Psi} \quad (52)$$
where \( \psi = e\psi/T_e, \quad \varphi = e\varphi/T_e \) and \( \lambda = k_{\perp} c_s^2/\omega_{ci}^2 \). Eqs (51) and (52) are valid for both kinetic Alfvén and ion acoustic waves. Note, however, since \((\omega_s, k_s)\) is an electrostatic mode, \( \varphi_s = \psi_s \), while for the kinetic Alfvén modes, \((\omega_k, k_k)\) and \((\omega_0, k_0)\), \( \varphi_{\perp 0} \neq \psi_{\perp 0} \).

In dealing with the nonlinear analyses, let us note that because the kinetic Alfvén waves have \( \lambda \sim 0(1) \), our results are, therefore, valid in the regime \( \lambda \gg \omega_0/\omega_{ci} \); while the classic magnetohydrodynamic results of Sagdeev and Galeev (1969) are valid in the opposite limit. The details of nonlinear interactions are different for the different modes as well as species and will be treated separately.

Let us first consider the ion acoustic \((\omega_s, k_s)\) mode. Since we are interested in either the resonant decay to the ion acoustic mode or the induced scattering decay when this mode is heavily damped, only nonlinear perturbations up to \( O(1/e) \) need to be kept here; i.e., we only have to calculate \( f_{\perp}^{(2)} \). For the electrons, \( |\omega_s|, |\omega_0| \ll \omega_{ce} \) and \( |k_{\perp} \rho_e| \ll |k_{\perp} \rho_t| < 1 \), so that \( \psi_{\perp} \) is negligible. The dominant nonlinear contributions to \( f_{\perp}^{(2)} \) is then found to originate from, referring to eq. (49),

\[
\nabla_\perp \left[ f_e^{(1)}(\Omega_0) f_{\parallel}^{(1)}(\Omega_+) + (0 \leftrightarrow -) \right] \text{term as well as the } \left[ (q/m)_e \left[ v_E(\Omega_0) \times B_\perp(\Omega_-) \right] \cdot \mathbf{E}_x + (0 \leftrightarrow -) \right] \left[ \partial f_e^{(0)} / \partial v_x \right] \text{term. Here, we have denoted } \Omega_0 = (\omega_0, k_0), \quad \Omega_- = (\omega_-, k_-) \text{ and } \Omega_+ = (\omega_+, k_+). \quad \text{With } |\omega_s| \ll |k_{\perp} v_T e|, \quad f_e^{(2)}(\Omega_+) \text{ is given by}
\]

\[
f_e^{(2)}(\Omega_+) \simeq f_e^{(0)} \left[ \psi_B(\Omega_+) + \psi_E(\Omega_+) \right],
\]

where \( \psi_B \) corresponds to the \( v_B f_e^{(1)} \) term:

\[
\psi_B(\Omega_+) = \frac{ie_s^2}{2\omega_{ci} k_{\parallel}} (k_\parallel \times k_0) \cdot e_x \left[ \frac{k_{\parallel} \lambda (1+\lambda_0)}{\omega_0} - \frac{k_{\parallel} \lambda (1+\lambda_0)}{\omega_-} \right] \varphi_{\perp 0},
\]

and \( \psi_E \) corresponds to the \( v_E \times B_\perp \) term:

\[
\psi_E(\Omega_+) = \frac{ie_s^2}{2\omega_{ci} k_{\parallel}} (k_\parallel \times k_0) \cdot e_x \left[ \frac{k_{\parallel} \lambda (1+\lambda_0)}{\omega_0} - \frac{k_{\parallel} \lambda (1+\lambda_0)}{\omega_-} \right] \varphi_{\perp 0}.
\]

In deriving eqs (54) and (55), we have made use of the fact that \( \Omega_0 \) and \( \Omega_- \) are kinetic Alfvén waves and, hence, \( \psi \) (or \( E_x \)) and \( B_\perp \) are related to \( \varphi \) by eq (16).

In treating the ions, we note first that because \( |\omega_i|, |\omega_0| \gg |k_{\parallel} v_T I|, |k_{\parallel} v_T T| \), \( v_B \) has negligible contribution and, from eq. (52), \( f_i^{(1)} \) of the kinetic Alfvén waves is reduced to

\[
f_i^{(1)}(\Omega) = - \tilde{\lambda} f_i^{(0)}(\Omega) + \frac{k_{\parallel} v_T}{\omega} f_i^{(0)}(\Omega) \frac{T_e}{T_i} \tilde{\psi}(\Omega) + f_{i\perp}^{(1)}(\Omega), \quad \text{for } \Omega = \Omega_0, \Omega_-.
\]

Secondly, we note that \( v_{pi} \) contains a nonlinear term from its convective part; i.e.,

\[
v_{pi}^{(2)} = (m/qB_0^2) (v_\parallel \cdot \nabla), \quad E_\perp. \quad \text{However, it can be shown that the contribution to the } \nabla_\perp \left[ v_\perp f \right] \text{term from } v_{pi}^{(2)} \quad f_{i\perp}^{(1)} \text{ cancels to the order } \omega_0/\omega_{ci} \text{ with that from}
\]
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Thus, the only net contribution comes from the $v_E f_{\perp}^{(1)}$ term; which corresponds to the usual $(v_E \cdot \nabla) v_z$ convective nonlinear (ponderomotive) force term in the parallel (to $B_0$) equation of motion. Another important nonlinear contribution comes from the $(v_E \times B_\perp) \cdot \hat{e}_z$ term, similar to that of the electrons. Combining, these nonlinear terms, we obtain

$$f_{i}^{a}(\Omega_{s}) \simeq c_s^2 \frac{\partial f_{i}^{(0)}}{\partial v_z} \left[ \tilde{\psi}_c (\Omega_{s}) + \tilde{\psi}_E (\Omega_{s}) \right],$$

(57)

where $\tilde{\psi}_c$ corresponds to the $\nabla \cdot [v_E f_{\perp}^{(1)}]$ nonlinearity and is given by

$$\tilde{\psi}_c (\Omega_{s}) = -i \frac{c_s^2}{2 \omega_c k_{ss}} (k_s \times k_0) \cdot \hat{e}_z \left( \frac{k_{zo} \lambda_0}{\omega_0} - \frac{k_{zd} \lambda_+}{\omega_-} \right) \varphi \varphi_0.$$

and $\tilde{\psi}_E (\Omega_{s})$ is given in eq. (55). Substituting eqs (51), (52), (53) and (57) into the quasineutrality equation

$$[n_e^{(1)}(\Omega_{s}) + n_e^{(2)}(\Omega_{s})] = [n_i^{(1)}(\Omega_{s}) + n_i^{(2)}(\Omega_{s})],$$

(58)

we have

$$e_s (\Omega) \varphi_s = \Lambda_s \left[ F(\Lambda) + e_s (\Omega_0) \right] \varphi \varphi_0,$$

(59)

where

$$e_s = 1 + \lambda_s - \chi_i,$$

(60)

$$\chi_i = c_s^2 \int \frac{\partial f_i^{(0)}}{\partial v_z} d v_z,$$

(61)

$$\Lambda_s = \frac{\Lambda}{k_{ss}} \left( \frac{k_{z0}}{\omega_0} - \frac{k_{zd}}{\omega_-} \right) \simeq \frac{\Lambda}{\omega_0},$$

(62)

$$\Lambda = -i c_s^2 \frac{(k_s \times k_0) \cdot \hat{e}_z}{2 \omega_c},$$

(63)

and

$$F(\Lambda) = \lambda_0 + \lambda_- + \lambda_0 \lambda_- - \lambda_s.$$

(64)

Eq. (60) describes the coupling of the ion acoustic mode to the lower-sideband kinetic Alfvén wave via the pump.

Let us now consider the kinetic Alfvén ($\omega_-, k_-$) mode. For this mode we have to calculate both the charge and parallel current density perturbations to $O(\varphi \varphi_0^2)$; i.e., $f_j^{(3)}(\Omega_s)$ in order to take into account the induced scattering process.

Thus, nonlinear contributions due to both $f_j^{(1)}(\Omega_s)$ and $f_j^{(2)}(\Omega_s)$ must be included. Let us first consider the electrons. Again, $v_{pe}$ has negligible contribution and dominant nonlinear contributions come from the $v_{E\perp} (\Omega_0) [f_e^{(1)}(\Omega_s) + f_e^{(2)}(\Omega_s)]$ term as well as the $[v_E (\Omega_0) \times B_\perp] \cdot \hat{e}_z$ term. It turns out, however, that $n_e^{(2)}$ is negligible.
compared to \( n_i^{(3)} \) due to the cancelling of dominant nonlinear terms; while \( n_e^{(3)} \) is given by

\[
 n_e^{(3)}(\Omega) \approx i \varepsilon \frac{2}{k_{z0}} (1+\lambda) k_z \boldsymbol{k}_x \cdot \mathbf{e}_z \, \mathcal{P}_0^\ast \, n_i^{(3)}(\Omega) .
\] (65)

Perturbations in parallel current density can then be obtained from the continuity eq:

\[
 J_{ze}^{(2)}(\Omega) \approx e k_{z-} \cdot \mathbf{v}_E(\Omega) n^* (\Omega) \mathbf{v}_E (\Omega) + \mathbf{v}_E (\Omega) n^{(1)} (\Omega) \mathbf{v}_E (\Omega) \mathbf{v}_E (\Omega) \mathbf{v}_E (\Omega) / k_{z-} ,
\] (66)

and

\[
 J_{ze}^{(3)}(\Omega) \approx e [k_{z-} \cdot \mathbf{v}_E^\ast (\Omega) n^{(1)} (\Omega) - \omega_n n^{(3)} (\Omega) \mathbf{n} / k_{z-} .
\] (67)

For the ions, because \( |\omega_-| \gg |k_z v_{TI}| \), the dynamics is mainly in the perpendicular (to \( \mathbf{B}_0 \)) plane. Thus, we can neglect \( J_{zi} \) and, from the continuity equation, we find the ion density perturbations to be

\[
 n_i^{(2)}(\Omega) \approx k_{z-} \cdot [N_0 v_{pl}^{(2)} + v_E (\Omega) n^* (\Omega) + v_E (\Omega) n^{(1)} (\Omega) / \omega_-]
\] (68)

and

\[
 n_i^{(3)}(\Omega) \approx k_{z-} \cdot [v_E^\ast (\Omega) n^{(1)} (\Omega) / \omega_- .
\] (69)

Here, \( v_{pl}^{(2)}(\Omega) \) is the nonlinear ion polarization drift due to the convective \( \mathbf{v}_\perp \cdot \nabla \mathbf{v}_\perp \) term.

The two field equations, eqs (3) and (4), including nonlinear perturbations become at \( \Omega = \Omega_- \)

\[
 n_e^{(1)} + n_e^{(2)} + n_e^{(3)} = n_i^{(1)} + n_i^{(2)} + n_i^{(3)} ,
\] (70)

\[
 \partial \frac{\nabla \cdot \mathbf{e}_z}{\nabla \omega_\perp} (\varphi - \psi) = \mu_0 \sum_{j = e, i} \partial \frac{\nabla \cdot \mathbf{J}_z^{(1)} + \mathbf{J}_z^{(2)} + \mathbf{J}_z^{(3)}},
\] (71)

which can be reduced to a single eq.

\[
 \varepsilon_A (\Omega) \mathcal{P}_- = \frac{1}{N_\varphi \omega_- (1+\lambda)} \left[ \omega_- (n_-^{(2)} + n_-^{(3)})
\right.
\]

\[
 - (1+\lambda) k_z \left( \mathbf{J}_z^{(2)} + \mathbf{J}_z^{(3)} \right) / \varepsilon \right] ,
\] (72)

where

\[
 \varepsilon_A (\Omega) = 1 - \frac{k_{z0}^2 \varphi_A^2 (1 + \varphi_A)}{1} ,
\] (73)

\[
 n_-^{(3)} - n_i^{(2)} + n_i^{(3)} = |n_i (\Omega) - n_e (\Omega)|^{(2)} + n^{(3)} ,
\] (74)

and

\[
 \mathbf{J}_z^{(2)} + \mathbf{J}_z^{(3)} \approx \mathbf{J}_z^{(2)} (\Omega) .
\] (75)
In deriving eq. (72), we have noted that $\Omega_-$ is a resonant kinetic Alfvén mode and, hence, $|\varepsilon_A| \ll 1$. Substituting eqs (65) to (69) for the nonlinear charge density and parallel current density perturbations into eq. (72), it reduces to

$$\tilde{\varepsilon}_A (\Omega_-) \tilde{\varphi}_- = \Lambda_A F(\tilde{\lambda}) \tilde{\varphi}_0 \tilde{\varphi}_s$$

where

$$\tilde{\varepsilon}_A = \varepsilon_A - \varepsilon_A^{(3)}; \quad (77)$$

$\varepsilon_A^{(3)}$ is due to third order perturbations

$$\varepsilon_A^{(3)} = |\tilde{\varphi}_0|^2 \Lambda \left(1 + \frac{k_{z0}\omega_-(1 + \tilde{\lambda}_0)}{k_z\omega_0} \right) \left(1 - \frac{k_{z0}\omega_-(1 + \tilde{\lambda}_0)}{k_z\omega_0} \right) \left(\frac{k_{z0}}{\omega_0} - \frac{k_z}{\omega_-}\right); \quad (78)$$

and

$$\Lambda_A = \Lambda/[(\omega_-\tilde{\lambda}_-)(1 + \tilde{\lambda}_-)] \simeq -\Lambda/[(\omega_0\tilde{\lambda}_0)(1 + \tilde{\lambda}_0)]. \quad (79)$$

Combining the two coupled equations, eqs (59) and (76) we derive the following dispersion relation for the parametric decay instabilities:

$$\left(\varepsilon_A - \varepsilon_A^{(3)}\right) \varepsilon_s = \Lambda_A \Lambda_s \frac{F(\tilde{\lambda}) \tilde{\varphi}_0}{\omega_0} \tilde{\varphi}_s^2; \quad (80)$$

where

$$\varepsilon_A^{(3)} = \varepsilon_A^{(3)} + \Lambda_A \Lambda_s \frac{F(\tilde{\lambda}) \tilde{\varphi}_0}{\omega_0} \tilde{\varphi}_s^2 \quad (81)$$

With $T_e \gg T_i$, the acoustic wave is weakly damped and we have the resonant decay instability. In this case, $\varepsilon_A^{(3)}$ can be neglected. Let $\omega_s = \omega_{sr} + iy$ and $\omega_- = -\omega_A + iy$, where $\omega_{sr}$ and $\omega_A = \omega_0 - \omega_{sr}$ satisfy, respectively, the dispersion relations for the ion acoustic and kinetic Alfvén waves. Equation (80) then reduces to

$$\frac{\partial \varepsilon_{sr}}{\partial \omega_{sr}} \frac{\partial \varepsilon_A}{\partial \omega_A} (\gamma + \Gamma_A (\gamma + \Gamma_s) = \left|\frac{\Lambda}{\omega_0} \frac{F(\tilde{\lambda})^2 |\tilde{\varphi}_0|^2}{\tilde{\lambda}_-(1 + \tilde{\lambda}_-)}, \quad (82)$$

with

$$\frac{\partial \varepsilon_{sr}}{\partial \omega_{sr}} = \frac{2(1 + \tilde{\lambda}_s)}{\omega_{sr}}, \quad (83)$$

$$\frac{\partial \varepsilon_A}{\partial \omega_A} = \frac{2}{\omega_A}, \quad (84)$$

and $\Gamma_A, \Gamma_s$ are the corresponding linear damping rates. From eq (82), we can deduce the threshold pump field by putting $\gamma = 0$. Well above the threshold field, the growth rate is given by

$$\gamma_D \simeq \frac{\omega_{cl}}{4} \left(\frac{\omega_{sr}}{\omega_A}\right)^{1/2} \frac{B_{\perp 0}}{B_0} \beta^{-1/2} \frac{|\tilde{F}(\lambda) \sin \theta|}{[(1 + \tilde{\lambda}_0)(1 + \tilde{\lambda}_-)(1 + \tilde{\lambda}_s)]^{1/2}}. \quad (85)$$
In deriving eq. (85), we have let \((k \times k_0) \cdot e_z = (k_\perp \times k_0) \cdot e_z = k_\perp k_\perp_0 \sin \theta\) and used the relation between \(B_{\perp 0}\) and \(\varphi_0 = (e \varphi_0 / T_e)\) expressed in eq. (16). The growth rate obtained here is larger than that of the ideal magnetohydrodynamic results due to Sagdeev and Galeev (1969) by a factor of \(\widetilde{\lambda} \omega_{ci} / \omega_A\). This enhancement is expected because, due to the finite \(\widetilde{\lambda}\)'s, nonlinear effects induced by the \(E \times B_0\) drifts of electrons and ions do not cancel each other to the order of \(\widetilde{\lambda}\) in the case considered here; while only ion polarization drift (which is smaller than the \(E \times B_0\) drift by a factor \(\left(\frac{\omega_A}{\omega_{ci}}\right)\)) contributes in the ideal magnetohydrodynamic limit. Furthermore, unlike the magnetohydrodynamic case in which only the backscattering is allowed, three different types of decay are possible here as illustrated in the \(\omega - k_z\) diagram in figure 2. Note also since \(F(\widetilde{\lambda}) \rightarrow 0\) as \(\widetilde{\lambda}_0 \rightarrow 0\), this decay process is pertinent to the pump wave being a kinetic Alfvén wave.

Let us now consider the case with \(T_e < 5T_i\) such that the ion acoustic wave is heavily Landau damped by ions. In this case, the decay instability is made through non-linear ion Landau damping, i.e., it is an ion-induced scattering process or,

\[\begin{align*}
\lambda_0 &\lesssim \lambda_A \\
\lambda_0 > \lambda_A &
\end{align*}\]

\((a)\) \hspace{2cm} \((b)\)

**Figure 2.** Three types of decay are possible from frequency \(\omega_0\) to \(\omega_A\) depending on the sign of \(k_z\) and \(K_{zA}\) as well as the size of \(\lambda_0\) \((=K_{zA} c_i^2/\omega_{ci}^2)\) with respect to \(\lambda_A\) \((=k_{zA} c_i^2/\omega_{ci}^2)\). In the magnetohydrodynamic limit where \(\lambda_0 = 0\), only case (c) is acceptable. In case (b), \(K_{zA}\) is decreased while in case (d) \(k_\perp\) is increased in consequence of the decay.
sometimes also called, a quasimode decay instability. From eq (80), the growth rate \( \gamma_N \) is obtained to be

\[
\gamma_N = -\frac{\omega_A}{2} \text{Im} \left[ \frac{A_A A_s F^2(\lambda)}{\epsilon_s} \text{Im} \left( \frac{\tilde{\epsilon}_A^{(3)}}{\epsilon_s} \right) \right] - \Gamma_A. \tag{86}
\]

Note, from eq (81), \( \tilde{\epsilon}_A^{(3)} \) does not contribute to the growth rate and \( \gamma_N \) further reduces to

\[
\gamma_N = \frac{\omega_{ci}}{8} \beta^{-1} \left| \frac{B_{\perp 0}}{B_0} \right|^2 \left( \frac{\omega_{ci}}{\omega_A} \frac{F^2(\lambda)}{1 + \lambda_0} \right) \text{Im} \left( \frac{\epsilon_s \sin \theta}{\epsilon_s} \right) - \Gamma_A, \tag{87}
\]

which has its maximum value at \( |\omega_s| \simeq k_{zz} v_T i \) and

\[
(\gamma_N)_{\text{max}} \simeq \frac{\omega_{ci}}{8} \beta^{-1} \left| \frac{B_{\perp 0}}{B_0} \right|^2 \left( \frac{\omega_{ci}}{\omega_A} \frac{F^2(\lambda)}{1 + \lambda_0} \frac{T_e/T_i}{(1 + \lambda_0)(1 + \lambda_0) + (T_e/T_i)^2} \right) - \Gamma_A. \tag{88}
\]

Again, we remark that (i) the growth rate obtained here is larger than the classic magnetohydrodynamic value by a factor \( (\lambda_0/\omega_{ci}/\omega_A)^2 \), (ii) our results are pertinent to the pump wave being a kinetic Alfvén wave and (iii) similar to the resonant decay instability, three types of decay are possible here (c.f. figure 2). Note that the threshold pump field depends on \( \Gamma_A \) which, in the collisionless regime, is mainly due to electron Landau damping and, typically, \( \Gamma_A/\omega_{ci} \sim O(10^{-2}) \). Then for a reasonable choice of other parameters; such as \( \beta \sim 10^{-2} \), \( \omega_{ci}/\omega_A \sim 10 \) and \( T_e \sim T_i \), the threshold amplitude of \( B_{\perp 0} \), \( (B_{\perp 0})_{\text{th}}/B_0 \sim 10^{-3} \).

3.3. Solitary wave of the kinetic Alfvén wave

In this section we introduce an exact nonlinear wave solution of the kinetic Alfvén wave. Solitary wave is a nonlinear wave which propagates in a stationary form. Usually, such a stationary wave form is generated by a balance between the nonlinear steepening effect and the dispersion effect. Several attempts have been made to determine the nonlinear structure of the Alfvén wave in a one-dimensional form propagating in the direction perpendicular (Adam and Allen 1958), parallel (Shaffman 1961), and oblique (Kazantsev 1963; Kellog 1964) to an ambient stationary magnetic field. The dispersion effect is introduced by the effect of finite frequency and electron inertia (Kakutani 1974). These solutions are obtained for a small amplitude wave in a cold plasma.

The kinetic Alfvén wave, which is essentially a hot plasma wave, has a dispersion for an oblique propagation. The nonlinearity originates from its compressible nature.

We make two assumptions. Firstly, that the plasma \( \beta(=2\mu_e nT/B_0^2) \) is small but much larger than the electron to ion mass ratio. Secondly, the variation exists in the \( x-z \) plane, where \( z \) is the direction of an ambient magnetic field \( B_0 \) and \( x \) is a perpendicular direction.

The low \( \beta \) assumption allows one to use the two potential fields \( \varphi \) and \( \psi \) to describe the electric field, \( E_x = -\partial \varphi/\partial x \) and \( E_z = -\partial \psi/\partial z \). They produce only shear perturbations in the magnetic field; \( B_x = B_0 \) (const), \( B_z = 0 \) and

\[
\frac{\partial B_y}{\partial t} = \frac{\partial^2}{\partial x \partial z} (\varphi - \psi). \tag{89}
\]

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The relevant field equations are the quasi-neutrality relation, for ion and electron densities \( n_i \) and \( n_e \),

\[
n_i = n_e, \tag{90}
\]

and Ampère's law for \( J_z \), the current density in the \( z \) direction,

\[
\frac{\partial^4}{\partial x^2 \partial z^2} (\varphi - \psi) = \mu_0 \frac{\partial^2 J_z}{\partial t \partial z}. \tag{91}
\]

In a low \( \beta \) plasma, the ion density is given by

\[
\frac{\partial n_i}{\partial t} - \frac{\partial}{\partial x} \left( \frac{m_i}{eB_0^2} \frac{\partial^2 \varphi}{\partial x \partial t} n_i \right) = 0, \tag{92}
\]

while the electron density is

\[
n_e = n_0 \exp \left( \frac{e\varphi}{T_e} \right), \tag{93}
\]

where \( T_e \) is the electron temperature. In eq. (92), the contribution of the parallel inertia term is ignored by the low \( \beta \) assumption. The Boltzman distribution for electron in eq. (93) is chosen here for convenience.

Because of the low \( \beta \) assumption, the contribution of ions to the current density \( J_z \) is negligible, hence \( J_z \) is given by the electron density,

\[
\frac{\partial J_z}{\partial z} = \frac{\partial (en_e)}{\partial t}. \tag{94}
\]

Eqs (91) to (94) become dimensionless if we introduce new variables, \( \xi = x/\rho_s \), \( \zeta = z/\rho_p \), \( \tau = \omega_{ci} t \), \( n = n/n_0 \), \( \varphi = e\varphi/T_e \), and \( \psi = e\psi/T_e \) where \( \rho_s = (T_e/m_i)^{1/2}/\omega_{ci} \), \( \omega_{ci} = eB_0/m_i \), \( \rho_p = (e^2n_0/e^2m_i)^{1/2} \); \( n_0 \) is the unperturbed plasma density and \( c \) is the speed of light. We then consider a one-dimensional simple wave solution propagating obliquely with respect to the ambient magnetic field. The simple wave is chosen to be stationary with respect to a coordinate \( \eta \), which is defined by

\[
\eta = K_x \xi + K_z \zeta - \tau. \tag{95}
\]

For the boundary conditions, we require that the number density perturbation \( n(\eta) \) as well as its derivatives, \( \partial n/\partial \xi, \partial n/\partial \zeta \) vanish as \( \xi \to \pm \infty \) and \( \zeta \to \pm \infty \). In terms of \( \varphi \) and \( \psi \), this means \( \varphi = \partial \varphi/\partial \xi = \partial \varphi/\partial \zeta = \partial^2 \varphi/\partial \xi^2 = \partial^2 \varphi/\partial \zeta^2 = 0 \) as \( \xi \to \pm \infty \) and \( \zeta \to \pm \infty \).

Using these boundary conditions, and the variable \( \eta \) eqs (91) and (92) can be integrated. If we then eliminate \( \varphi, \psi \) and \( J_z \) from eqs (91) to (94), we have the following nonlinear wave equation,

\[
(K_x^2 - \tilde{n}) (1 - \tilde{n}) + K_x^2 K_z^2 \tilde{n} \frac{d}{d\eta} \left( \frac{1}{\tilde{n}} \frac{d\tilde{n}}{d\eta} \right) = 0. \tag{96}
\]

The linear dispersion relation is obtained by substituting \( \tilde{n} = 1 + \delta n \exp (iK\eta) = \exp iK(K_x \xi + K_z \zeta - \tau) \equiv \exp i(k_x \chi + k_z \zeta - \omega t) \);

\[
1 = K_x^2 (1 + K^2 K_x^2). \tag{97}
\]
In terms of the original coordinates,
\[ \omega^2 = k_z^2 v_A^2 (1 + k_x^2 \rho_s^2). \] (98)

This is the dispersion relation of the kinetic Alfvén wave. The related solitary wave solution is expected to have a structure scaled by the ion gyroradius \( \rho_s \) in the perpendicular direction and to propagate almost in the parallel direction.

Eq. (96) can be readily integrated once by changing the independent variable from \( \eta \) to \( \bar{n} \). Using the condition \( dn/d\eta = 0 \) at \( \bar{n} = 1 \), we obtain
\[
\left( \frac{dn}{d\eta} \right)^2 = \frac{2\bar{n}}{K_x^2 K_z^2} [(1 - \bar{n}) (K_z^2 + \bar{n}) + (1 + K_z^2)\bar{n} \ln \bar{n}]
\]
\[\equiv \frac{F(\bar{n}, K_z^2)}{K_x^2 K_z^2}. \] (99)

Expanding \( F \) around \( \bar{n} = 1 \) identifies \( \bar{n} = 1 \) as a double root, a condition required for a solitary wave.

The potential function \( F(\bar{n}, K_z^2) \) is plotted in figure 3 for various values of \( K_z^2 \). The nonlinear wave(s) that we are seeking exist only for the range of density which

![Figure 3. The nonlinear potential function \( F(\bar{n}, K_z^2) \) is shown for various values of \( K_z^2 \).](image-url)
makes \( F \geq 0 \) including \( \bar{n} = 1 \). We see from this figure that there exists no range if \( K_x^2 < 1 \), only one range, \( 0 \leq \bar{n} \leq 1 \), if \( K_x^2 = 1 \), and two ranges, \( 0 \leq \bar{n} \leq 1 \) and \( 1 \leq \bar{n} \leq \bar{n}_{\text{max}} \), if \( K_x^2 > 1 \). \( K_x = 1 \) corresponds to a wave that propagates at the Alfvén speed in the \( z \) direction, hence \( K_x^2 > 1 (K_x^2 < 1) \) corresponds to a sub-Alfvénic (super-Alfvénic) wave. For the sub-Alfvénic range there are two types of solitary wave solutions, one accompanying a density hump \( (\bar{n} > 1) \) and the other density dip \( (\bar{n} \leq 1) \). Qualitatively this result agrees with the small amplitude result. However, the wave corresponding to \( \bar{n} \leq 1 \) has a minimum density of zero which is beyond the scope of our theoretical framework (since the Alfvén speed becomes infinity), and other effects, such as the finite frequency effect, need to be introduced to obtain a reasonable solution in this range of \( \bar{n} \).

Consequently, we consider only the wave corresponding to \( \bar{n} \geq 1 \). If \( K_x^2 \) deviates from unity by a small amount, say \( M = K_x^2 - 1 \ll 1 \), \( (M > 0) \), the maximum amplitude \( \bar{n}_{\text{max}} \) of the solitary wave is correspondingly small. For such a case we can integrate eq. (99) analytically for a small density perturbation, \( \Delta = \bar{n} - 1 \), to give

\[
\Delta = \frac{3M}{2} \text{sech}^2 \left( \frac{\sqrt{M} \eta}{2 |K_x K_z|} \right). \tag{100}
\]

The corresponding magnetic field perturbation \( B_y \) (normalized by \( B_0 \)) is obtained from eq. (89);

\[
\frac{B_y}{B_0} = \bar{B}_y = -\frac{c_s}{v_A} \int (\bar{n} - 1) \frac{d\eta}{K_x K_z} \\
\simeq -3 \sqrt{M} \text{sgn} (K_x K_z) \frac{c_s}{v_A} \tanh \left( \frac{\sqrt{M} \eta}{2 |K_x K_z|} \right). \tag{101}
\]

The wave accompanies a finite \( x \) component of the electric field as \( \eta \to \pm \infty \), which is given by \( E_x (\eta \to \pm \infty) = v_A K_x^{-1} B_y (\eta \to \pm \infty) \). In general \( E_x \) is given by

\[
\bar{E}_x = \frac{E_x}{V_A B_0} = -\frac{c_s}{v_A} \int \left( \frac{\bar{n} - 1}{\bar{n}} \right) \frac{d\eta}{K_x}. \tag{102}
\]

The present solution is exact to an arbitrary amplitude as long as the result is consistent with the low \( \beta \) assumption. This means that the effective phase velocity of the wave should be larger than the ion sound speed \( c_s \). Hence the maximum allowable value of \( K_x \) is \( v_A/c_s \), or \( K_x^2 \leq \beta^{-1} \). For a large value of \( K_x^2 \), the peak density \( \bar{n}_{\text{max}} \) is related to \( K_x^2 \) through \( K_x^2 \approx \bar{n}_{\text{max}} / \ln \bar{n}_{\text{max}} \). Hence the present result should be valid if \( \bar{n}_{\text{max}} \leq \beta^{-1} \).

Figure 4 shows the density profiles for \( \bar{n}_{\text{max}} = 10 \) and \( \bar{n}_{\text{max}} = 5 \) which are obtained by numerically integrating eq. (100). The dotted lines show the sech\(^2\) solutions that are fitted to the maximum density point. The exact solitary wave is seen to have a wider structure than the ideal sech\(^2\) solution. The corresponding values \( \bar{B}_y \) and \( \bar{E}_x \) are shown in figure 5. The integration constant is chosen so that these values vanish at the origin of the \( \eta \) coordinate. As expected they have kink structures.
Kinetic properties of Alfvén waves

Figure 4. Normalized number density $\bar{n}$ of the solitary wave for two values of $K_z^2$. Dotted curves are $\text{sech}^2$ function of eq. (100) fitted at $\bar{n} = \bar{n}_{\text{max}}$.

Figure 5. Magnetic field $\bar{B}_y v_A / C_s$ and electric field $\bar{E}_x v_A / C_s$, $\bar{B}_y v_A / C_s$, $\bar{E}_x v_A / C_s$ ($\bar{n}_{\text{max}} = 10$) of the solitary wave for $K_z^2 = 4.78$ ($\bar{n}_{\text{max}} = 10$).
Figure 6. The magnetic line of force associated with the solitary wave is shown by the solid curve with arrows. It lies on the edge of an oblique slice (shown by the hatched area) of a triangular shaped roof. The field line exists in the $y$-$z$ plane while it varies only in the $\eta$ direction which is in the $x$-$z$ plane. At a fixed time, $\eta = K_x \omega_e X'(c^e)^2 + K_x \omega_p z'/c$.

The structure of the total magnetic line of force is also interesting. Because $B_z(\eta \to \pm \infty) \to \pm$ constant, the line of force must be straight as $\eta \to \pm \infty$. Hence the projection of the field line into the $\eta$-$y$ plane is basically a triangular shape with a round top as shown in figure 6. The actual line of force exists in $y$-$z$ plane. Because the phase stationary line at $t = 0$ exists on $\eta = 0$, such a line can be expressed by $\omega_p z'/c = -(K_x/K_\eta)x/\rho_a$ namely $z = -(K_x/K_\eta)(\nu_d/c_e)x$. Hence the $z$ axis has an angle $\theta = -\tan^{-1}(K_x/c_e(\nu_d/c_e))$ with respect to $\eta = 0$ line. Therefore, the actual line of force follows the edge of an oblique slice of the triangular roof-like surface by the $y$-$z$ plane as shown by the arrowed solid curve in the figure. For a linear wave, $K_x$ approximately equals $\omega_e/\omega \gg 1$, as can be seen from eq. (100). Hence, the angle $\theta$ is usually very small.

The stability of the solution against nonlinear decay to the ion acoustic wave of the previous section, against the linear two stream instability and against the linear hydromagnetic Kelvin-Helmholtz instability (Chandrasekhar 1961) has been studied. We find the solution to be stable as long as the ion temperature is comparable to the electron temperature.

4. Application to plasma heating

In this section we consider the use of the kinetic Alfvén wave for supplementary heating of a Tokamak type plasma. As the ohmic heating saturates at a temperature near 1 keV, a supplementary heating is needed to increase the temperature to a fusion temperature of 10 keV. Several methods have been proposed for this purpose, but none is proved conclusively to be the best choice. A reactor Tokamak typically has a dimension of the following size.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>density of electrons</td>
<td>$n_e, 2 \times 10^{14}$</td>
</tr>
<tr>
<td>toroidal field $B_0$</td>
<td>$70$ kG</td>
</tr>
<tr>
<td>minor radius $a$</td>
<td>$3$ m</td>
</tr>
<tr>
<td>major radius $R$</td>
<td>$12$ m</td>
</tr>
</tbody>
</table>

This gives the plasma volume of approximately $2 \times 10^9 m^3$ and output power of $3$ GW. To heat such a plasma within the expected energy containment time of $10$ sec., one needs the input power of the supplementary heating of $200$ MW. If one substracts
heating efficiency, this number indicates that one needs a power source with an efficiency of the order of 50%. This is rather a severe requirement for a power source with 200 MW.

The Alfvén wave whose frequency range is on the order of 1 MHz is attractive in this respect because a power source of this magnitude and efficiency is currently available.

Plasma is heated when the kinetic Alfvén wave, which can be excited in the plasma through the resonant mode conversion as discussed in section 1, is absorbed by the plasma particles. Supplementary heating has meaning only in the collisionless regime where the ohmic heating can no longer operate. In this regime, Landau damping or nonlinear wave particle interaction is the important absorption process.

Since the linear Landau damping, which effectively operates only on electrons, is expected to saturate quickly due to the plateau formation of the distribution function, the nonlinear heating is probably the most important process. There, ions can be heated by the nonlinear ion Landau damping. The threshold value of the wave amplitude $B_\perp/B_0$ is given from eq. (87) as

$$v_{\text{th}} \leq 0.1 \frac{B_\perp^2 \omega_{ct}^2}{B_0^2 \omega}.$$  \hspace{1cm} (103)

Here we have assumed that the electron Landau damping has been saturated by the plateau formation, hence only the ion viscous damping $v_{\text{th}} \lambda_i$ is a dissipative process. The ion-ion collision frequency $v_{\text{th}}$ for 1 keV plasma with a density of $5 \times 10^{13}$ cm$^{-3}$ is approximately $10^3$ sec$^{-1}$. Hence the threshold amplitude is given by

$$B_\perp(G) \approx \sqrt{\omega/50}.$$  \hspace{1cm} (104)

If one uses 1 MHz, the threshold amplitude of the kinetic Alfvén wave becomes approximately 50 G. This is quite a reasonable value if we recall that the amplitude of the kinetic Alfvén wave is enhanced, due to the resonant mode conversion, by a factor of $(\rho_d a)^{-1/2} \sim 30$, compared with the externally applied field.

If the kinetic Alfvén wave dissipates completely as it propagates through the plasma column, the heating rate is given by the absorption rate of the externally applied oscillating field by the resonant absorption. This means that the damping rate per cycle should be larger than the wavelength ($\sim \rho_d$) over the minor radius of the column, which for the example shown here is $10^{-3}$. The absorption rate has been evaluated by Chen and Hasegawa (1974), as

$$\frac{dW}{dt} \approx V \frac{B_\perp(x_o)^2}{2\mu_o} \omega,$$  \hspace{1cm} (105)

where $V$ is the plasma volume and $B_\perp(x_o)$ is the value of the magnetic field at the location corresponding to the resonant surface in the absence of the plasma. This relation shows that to provide 200 MW to the plasma, one needs only 4 G of oscillating magnetic field. Inside the plasma, this field will be enhanced to about 100 G and will be well beyond the threshold value for the nonlinear Landau damping.
5. Conclusions

Kinetic properties of the Alfvén wave, which originate from the finite perpendicular wavelength effects, are shown for both linear and nonlinear regime. The wave accompanies a parallel electric field and can propagate in the direction perpendicular to the ambient magnetic field. The kinetic Alfvén wave can be excited in a plasma either by drift wave instability or resonant mode conversion of a surface magnetohydrodynamic wave.

The turbulent kinetic Alfvén wave is shown to produce anomalous transport phenomena for plasma particles. Anomalous cross $B$ diffusion and parallel viscosity coefficients are derived using quasilinear theory.

The nonlinear decay of the kinetic Alfvén wave is shown to take place more easily than the magnetohydrodynamic Alfvén wave. This property is shown to be important for the application to the plasma heating.

It is shown that the kinetic Alfvén wave admits an exact solitary wave solution which originates from the balance between the perpendicular dispersion and the compressible nonlinearity of the wave.

In the last section of this paper, applications of the kinetic Alfvén wave to the supplementary heating of a Tokamak plasma are discussed. It is shown that with an external oscillating magnetic field of a few Gauss, nonlinear heating of ions can be expected at a required rate of 200 MW for a reactor plasma.

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