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A remark on the paper of Dwivedi

P. K. BANERJI AND P. L. SETHI

Department of Mathematics, University of Jodhpur, Jodhpur

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ABSTRACT

In the present paper the method of Dwivedi and Trivedi in solving the dual integral equations, has been modified by presenting a very comprehensive and lucid solution, based on the analysis suggested by Fox.

1. INTRODUCTION

CERTAIN mixed boundary value problems may be reduced to the solution of dual integral equations of the type

$$\int_0^\infty \frac{1}{2}\sqrt{xu}\, G_{0,4}^{2,0}\left(\frac{x^2u^2}{16}\,\middle|\,\frac{\alpha}{2}, \frac{\beta}{2}, -\frac{\alpha}{2}, -\frac{\beta}{2}\right) f(u)\, du = g(x); \quad 0 < x < 1 \tag{1}$$

and

$$\int_0^\infty \frac{1}{2}\sqrt{xu}\, G_{0,4}^{2,0}\left(\frac{x^2u^2}{16}\,\middle|\,\frac{\gamma}{2}, \frac{\delta}{2}, -\frac{\gamma}{2}, -\frac{\delta}{2}\right) f(u)\, du = h(x); \quad x > 1 \tag{2}$$

where $g(x)$ and $h(x)$ are known functions, $G_{p,q}^{m,n}(x)$ is the Meijer's G-function as unsymmetrical Fourier kernel given by Narain,[1] α, β, γ, δ are parameters and $f(u)$ is to be determined. Recently Dwivedi and Trivedi[2] have given a solution of (1) and (2) by employing a method of antiquity given earlier by Titchmarsh.[3] A modern approach is presented in this note, following the analysis recently suggested by Fox.[4]

To simplify the analysis, from Erdélyi[5] (p. 353), it follows that,

$$M\left\{\frac{1}{2}\sqrt{xu}\, G_{0,4}^{2,0}\left(\frac{x^2u^2}{16}\right)\right\} = \frac{4^{s-\frac{1}{2}}\,\Gamma\left(\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\beta}{2}+\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\beta}{2}+\frac{3}{4}-\frac{s}{2}\right)}\, x^{-s}, \tag{3}$$

where M denotes the Mellin transform.

Fractional integration operators defined by Erdèlyi,[6] are

$$I[\mu, \nu;\ m;\ w(x)]$$

$$= \frac{m}{\Gamma(\mu)}\, x^{-\nu-m\mu+m-1} \int_0^x (x^m - v^m)^{\mu-1}\, v^{\nu-1}\, w(v)\, dv \tag{4}$$

and

$$R[\mu, \nu;\ n;\ w(x)]$$

$$= \frac{m}{\Gamma(\mu)}\, x^{\nu} \int_x^\infty (v^m - x^m)^{\mu-1}\, v^{-\nu-m\mu+m-1}\, w(v)\, dv, \tag{5}$$

provided

$$w(x) \in L_p(0, \infty), \quad p \geq 1, \quad u > 0 \quad \text{and} \quad \nu > \frac{1-p}{p}.$$

2. SOLUTION OF THE EQUATIONS

Let us write $M\{f(u)\} = F(s)$, use the result by Fox[4] [p. 391 (12)] to (1) and (2) and then by virtue of (3), we arrive at

$$\frac{1}{2\pi i} \int_C^\infty \frac{4^{s-\frac{1}{2}}\, \Gamma\left(\frac{\alpha}{2} + \frac{1}{4} + \frac{\delta}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{1}{4} + \frac{s}{2}\right) F(1-s)}{\Gamma\left(\frac{\alpha}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{3}{4} - \frac{s}{2}\right)}$$

$$\times\, x^{-s}\, ds = g(x), \qquad 0 < x < 1 \tag{6}$$

$$\frac{1}{2\pi i} \int_C^\infty \frac{4^{s-\frac{1}{2}}\, \Gamma\left(\frac{\gamma}{2} + \frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{1}{4} + \frac{s}{2}\right) F(1-s)}{\Gamma\left(\frac{\gamma}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{3}{4} - \frac{s}{2}\right)}$$

$$\times\, x^{-s}\, ds = h(x), \qquad x > 1 \tag{7}$$

Now we aim at the following transformations:

$$\left[\Gamma\left(\frac{\alpha}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{3}{4} - \frac{s}{2}\right)\right]^{-1}$$

into

$$\left[\Gamma\left(\frac{\gamma}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{3}{4} - \frac{s}{2}\right)\right]^{-1} \tag{8}$$

and

$$\left[\Gamma\left(\frac{r}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\delta}{2}+\frac{1}{4}+\frac{s}{2}\right)\right]$$

into

$$\left[\Gamma\left(\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\beta}{2}+\frac{1}{4}+\frac{s}{2}\right)\right]. \tag{9}$$

For brevity, we write

$$\Gamma\left[\left(\frac{\gamma}{2}-\frac{\alpha}{2}\right),\ 2\left(\frac{\alpha}{2}+\frac{3}{4}\right)-1:2:w(x)\right]=I_1[w(x)] \tag{10}$$

$$I\left[\left(\frac{\delta}{2}-\frac{\beta}{2}\right),\ 2\left(\frac{\beta}{2}+\frac{3}{4}\right)-1:2:w(x)\right]=I_2[w(x)] \tag{11}$$

$$R\left[\left(\frac{\gamma}{2}-\frac{\alpha}{2}\right),\ 2\left(\frac{\alpha}{2}+\frac{1}{4}\right):2:w(x)\right]=R_1[w(x)] \tag{12}$$

$$R\left[\left(\frac{\delta}{2}-\frac{\beta}{2}\right),\ 2\left(\frac{\beta}{2}+\frac{1}{4}\right):2:w(x)\right]=R_2[w(x)] \tag{13}$$

Now let us proceed to perform the first transformation on (6). Replace x by v, multiply by $v^{2(\alpha/2+3/4)-1}\,(x^2-v^2)^{(\gamma/2-\alpha/2)-1}$ and integrate w.r.t. v from 0 to x, and apply the beta function formula, we obtain

$$\frac{1}{2\pi i}\int_C \frac{4^{s-\frac{1}{2}}\,\Gamma\left(\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\beta}{2}+\frac{1}{4}+\frac{s}{2}\right)F(1-s)}{\Gamma\left(\frac{\gamma}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\delta}{2}+\frac{3}{4}-\frac{s}{2}\right)}\,x^{-s}\,ds$$

$$=\frac{2x^{2-2(\gamma/2+3/4)}}{\Gamma\left(\frac{\gamma}{2}-\frac{\alpha}{2}\right)}\int_0^x v^{2(\alpha/2+3/4)-1}\,(x^2-v^2)^{\gamma/2-\alpha/2-1}\,g(v)\,dv$$

$$=I_1[g(x)],\qquad 0<x<1. \tag{14}$$

Now we transform (14) by the application of the operator I_2 to obtain

$$\frac{1}{2\pi i}\int_C \frac{4^{s}\,\Gamma\left(\frac{\alpha}{2}+\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(\frac{\beta}{2}+\frac{1}{4}+\frac{s}{2}\right)F(1-s)}{\Gamma\left(\frac{\gamma}{2}+\frac{3}{4}-\frac{s}{2}\right)\Gamma\left(\frac{\delta}{2}+\frac{3}{4}-\frac{s}{2}\right)}\,x^{-s}\,ds$$

$$=2I_1\left[I_2\{g(x)\}\right],\qquad 0<x<1. \tag{15}$$

Similarly if we transform (7) by the application of the operators R_1 and R_2 respectively, we obtain

$$\frac{1}{2\pi i} \int_{c} \frac{4^s\, \Gamma\left(\frac{\alpha}{2} + \frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{1}{4} + \frac{s}{2}\right) F(1-s)}{\Gamma\left(\frac{\gamma}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{3}{4} - \frac{s}{2}\right)}\, x^{-s}\, ds$$

$$= 2R_1 \left[R_2 \{h(x)\}\right], \qquad x > 1. \tag{16}$$

Finally, on writing

$$t(x) = \begin{cases} 2I_1 \; [I_2 \; \{g(x)\}], & 0 < x < 1 \\ 2R_1 \, [R_2 \, \{h(x)\}], & x < 1 \end{cases} \tag{17}$$

we obtain from (15) and (16)

$$\frac{1}{2\pi i} \int_{c} \frac{4^s\, \Gamma\left(\frac{\alpha}{2} + \frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{1}{4} + \frac{s}{2}\right) F(1-s)}{\Gamma\left(\frac{\gamma}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{3}{4} - \frac{s}{2}\right)}\, x^{-s}\, ds = t\ (x) \tag{18}$$

The above relation (18) is, in fact, the reduction of (6) and (7) into two equations, but with a common kernel. Treating the kernel of (18) as the unsymmetrical Fourier kernel and following the analysis of Fox,[4] it can be written as

$$f(x) = \frac{1}{2\pi i} \int_{c} \frac{4^s\, \Gamma\left(\frac{\gamma}{2} + \frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{1}{4} + \frac{s}{2}\right) \tau(1-s)}{\Gamma\left(\frac{\alpha}{2} + \frac{3}{4} - \frac{s}{2}\right) \Gamma\left(\frac{\beta}{2} + \frac{3}{4} - \frac{s}{2}\right)} \times x^{-s}\, ds, \tag{19}$$

where

$$M\{t(x)\} = \tau(s).$$

This is the formal solution of (6) and (7) and many important properties of $f(x)$ are expected to result from it. Finally, applying the Parseval theorem[3], (19) reduces to

$$f(x) = \int_0^{\infty} G^{2,\,0}_{0,\,4}\left(\frac{x^2 u^2}{16} \middle| \frac{\gamma}{2} + \frac{1}{4},\ \frac{\delta}{2} + \frac{1}{4};\ -\frac{\beta}{2} + \frac{4}{1},\ -\frac{\alpha}{2} + \frac{1}{4}\right) \times t(u)\, du$$

$$= \int_0^1 G_{0,4}^{2,0}\left(\frac{x^2u^2}{16}\middle| \frac{\gamma}{2}+\frac{1}{4}, \frac{\delta}{2}+\frac{1}{4}; -\frac{\beta}{2}+\frac{1}{4}, -\frac{\alpha}{2}+\frac{1}{4}\right)$$

$$\times\ 2I_1\,[I_2\,\{g\,(u)\}]\,du$$

$$+ \int_1^\infty G_{0,4}^{2,0}\left(\frac{x^2u^2}{16}\middle| \frac{\gamma}{2}+\frac{1}{4}, \frac{\delta}{2}+\frac{1}{4}; -\frac{\beta}{2}+\frac{1}{4}, -\frac{\alpha}{2}+\frac{1}{4}\right)$$

$$\times\ 2R_1\,[R_2\,\{h'(u)\}]'du. \tag{20}$$

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