Proc. Indian Acad. Sci., Vol. 85 A, No. 4, 1977, pp. 179–185.

Triple and quadruple integral equations occurring in diffraction theory

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MS received 4 March 1974; in revised form 8 November 1976

ABSTRACT

A formal scheme has been set up to solve the triple and quadruple integral equations of certain type which occur in diffraction theory. The operational theory has been employed in the analysis and equations have been reduced to the solution of simultaneous Fredholm equations.

1. INTRODUCTION

HERE the equations under consideration are:

(i) The triple integral equations

$$\int_0^\infty \psi(u)\, J_\nu(xu)\, du = G_1(x), \quad (0 < x < a),$$

$$= 0, \qquad (b < x < \infty), \tag{1}$$

$$\int_k^\infty \{1 + H(u)\}\, u^{-\mu-\nu} (u^2 - k^2)^\beta J_\mu(xu)\, \psi(u)\, du = F_2(x), (a < x < b).$$

(ii) The quadruple integral equations

$$\int_0^\infty \psi(u)\, J_\nu(xu)\, du = G_2(x), \quad (a < x < b),$$

$$= 0, \qquad (c < x < \infty), \tag{2}$$

$$\int_k^\infty \{1 + H(u)\}\, u^{-\mu-\nu} (u^2 - k^2)^\beta \psi(u)\, J_\mu(xu)\, du = F_2(x), \ (0 < x < a),$$

$$\int_k^\infty \{1 + \bar{H}(u)\}\, u^{-\mu-\nu} (u^2 - k^2)^\beta \psi(u)\, J_\mu(xu)\, du = \bar{F}_3(x),$$

$$(b < x < c).$$

The above two sets of equations are extensions of the dual integral equations arising in diffraction theory considered by Burlak.[1] Lowndes[2] has recently applied his generalised Erdélyi-Köber operators to the equations considered by Burlak. Lowndes method is extended here to solve the sets of equations (1) and (2). The generalised Erdélyi-Köber operators due to Lowndes and their properties have been used and also two new operators have been introduced.

For $\mu = \nu$, $k = 0$ and $\beta = \mu - \alpha$ the set (1) reduces to the equations considered by Cooke[3] and for $\mu = \nu$, $k = 0$, $\psi(u) = \xi^{-2\delta}\phi(\xi)$ and $\beta = \nu - \alpha + \delta$ the set (2) reduces to those solved by Ahmad.[4] Their results can be directly deduced from the results obtained in the present paper.

2. THE OPERATORS

For ready reference we give here the operators and their properties used in the analysis. The generalised Erdélyi-Köber operators are[4]

$$\binom{x}{a} T_k(\eta, \alpha) f(x) = 2^{\alpha} x^{-2\eta-2\alpha} k^{1-\alpha} \int_a^x u^{1+2\eta} (x^2 - u^2)^{1/2(\alpha-1)} \times J_{\alpha-1}\{k\sqrt{(x^2-u^2)}\} f(u)\, du, \quad \alpha > 0, \tag{3}$$

$$= 2^{\alpha} x^{-1-2\eta-2\alpha} k^{-\alpha} \frac{d}{dx} \int_a^x u^{1+2\eta} (x^2 - u^2)^{\alpha/2} \times J_{\alpha}\{k\sqrt{(x^2-u^2)}\} f(u)\, du, \quad -1 < \alpha < 0, \tag{4}$$

$$\binom{b}{x} A_k(\eta, \alpha) f(x) = 2^{\alpha} x^{2\eta} k^{1-\eta} \int_x^b u^{1-2\eta-2\sigma} (u^2 - x^2)^{1/2(\alpha-1)} \times J_{\alpha-1}\{k\sqrt{(u^2-x^2)}\} f(u)\, du, \quad \alpha > 0, \tag{5}$$

$$= -2^{\alpha} x^{2\eta-1} k^{-\alpha} \frac{d}{dx} \int_x^b u^{1-2\eta-2\sigma} (u^2 - x^2)^{\alpha/2} \times J_{\alpha}\{k\sqrt{(u^2-x^2)}\} f(u)\, du, \quad -1 < \alpha < 0, \tag{6}$$

where $\eta > -\frac{1}{2}$. Also the operators $T_{ik}(\eta, \alpha)$ and $A_{ik}(\eta, \alpha)$ are defined by the above equations when $J_{\alpha-1}$ is replaced by $I_{\alpha-1}$.

The generalised operator of Hankel transform is

$$S\binom{a,\, b,\, k}{\eta,\, \alpha,\, \sigma} f(x) = 2^{\alpha} x^{2\sigma-\alpha} (x^2 - a^2)^{-\sigma} \int_k^{\infty} u^{1-2\sigma-\alpha} (u^2 - k^2)^{\sigma} f(u) \times J_{2\eta+\alpha}\{\sqrt{[(x^2-a^2)(u^2-b^2)]}\}\, du. \tag{7}$$

We shall make use of the basic properties of the above operators as given below

$$\binom{x}{a} T_{ik}^{-1}(\eta, \alpha) f(x) = \binom{x}{a} T_k(\eta + \alpha, -\alpha) f(x),$$

$$\binom{x}{a} T_k^{-1}(\eta, \alpha) f(x) = \binom{x}{a} T_{ik}(\eta + \alpha, -\alpha) f(x), \tag{8}$$

$$\binom{b}{x} A_{ik}^{-1}(\eta, \alpha) f(x) = \binom{b}{x} A_k(\eta + \alpha, -\alpha) f(x),$$

$$\binom{b}{x} A_k^{-1}(\eta, \alpha) f(x) = \binom{b}{x} A_{ik}(\eta + \alpha, -\alpha) f(x), \tag{9}$$

$$S^{-1}\begin{pmatrix}0, & k, & k\\ \eta, & \alpha, & \sigma\end{pmatrix} = S\begin{pmatrix}k, & & 0, 0\\ \eta + \alpha, & -\alpha, & \sigma\end{pmatrix}, \tag{10}$$

$$S\begin{pmatrix}0, & k, & k\\ \eta + \alpha, & \beta, & \sigma\end{pmatrix} S\begin{pmatrix}k, & 0, & 0\\ \eta, & \alpha, & \sigma - \eta - \tfrac{1}{2}\alpha\end{pmatrix} = T_k(\eta, \alpha + \beta), \tag{11}$$

$$S\begin{pmatrix}0, & 0, & 0\\ \eta, & \alpha, & \end{pmatrix} S\begin{pmatrix}k, & & 0, 0\\ \eta + \alpha, & \beta, & \eta + \alpha + \tfrac{1}{2}\beta\end{pmatrix} = A_{ik}(\eta, \alpha + \beta). \tag{12}$$

We shall also find it convenient to have expressions for integral operators of the type

$$\binom{x}{d} T_k^{-1}(\eta, \alpha) \binom{f}{e} T_k(\eta, \alpha) f(x) = \begin{pmatrix}x, & f\\ d, & e\end{pmatrix} L^*(\eta, \alpha) f(x), \tag{13}$$

$$x > d \geq f > e,$$

$$\binom{d}{x} A_{ik}^{-1}(\eta, \alpha) \binom{f}{e} A_{ik}(\eta, \alpha) f(x) = \begin{pmatrix}d, & f\\ x, & e\end{pmatrix} M^*(\eta, \alpha) f(x),$$

$$x < d \leq e \leq f. \tag{14}$$

Following Cooke[3] we have simplified the expressions for L^* and M^* in the form given below:

$$\begin{pmatrix}x, & f\\ d, & e\end{pmatrix} L^*(\eta, \alpha) f(x)$$

$$= \frac{2 \sin \alpha\pi}{\pi} x^{-2\eta} \int_e^f u^{1-2\eta} f(u) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{k}{2}\right)^{2(m+n)} (-1)^m}{(1-\alpha)_n (\alpha)_m}$$

$$\times \frac{(x^2 - d^2)^{\eta - \alpha} (d^2 - u^2)^{m + \alpha - 1}}{m!\, n!}$$

$$\times {}_2F_1\left(1 - \alpha - m, 1;\, n - \alpha + 1;\, \frac{d^2 - x^2}{d^2 - u^2}\right) du, \tag{15}$$

$$\begin{pmatrix} d, & f \\ x, & e \end{pmatrix} M^* (\eta, \alpha) f(x)$$

$$= \frac{2 \sin \alpha\pi}{\pi} x^{2\eta + 2\alpha} \int_e^f u^{1-2\eta-2\alpha} f(u) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{k}{2}\right)^{2(m+n)} \frac{(-1)^m}{(1-\alpha)_n (\alpha)_m}$$

$$\times \frac{(d^2 - x^2)^{\eta-\alpha} (u^2 - d^2)^{m+\alpha-1}}{m!\, n!}$$

$$\times \; {}_2F_1 \left(1 - \alpha - m, 1;\ n - \alpha + 1;\ \frac{d^2 - x^2}{d^2 - u^2}\right) du. \tag{16}$$

Also it can be shown easily that

$$\begin{pmatrix} d \\ c \end{pmatrix} T_k^{-1} \begin{pmatrix} x \\ c \end{pmatrix} T_k f(x) = - \begin{pmatrix} x, & d \\ d, & c \end{pmatrix} L^* f(x), (x > d > c), \tag{17}$$

$$\begin{pmatrix} e \\ d \end{pmatrix} A_{ik}^{-1} \begin{pmatrix} e \\ x \end{pmatrix} A_{ik}\ f(x) = - \begin{pmatrix} d, & e \\ x, & d \end{pmatrix} M^* f(x), (x < d < e). \tag{18}$$

Heie we have dropped the subscripts of the operators T_k, A_{ik}, L^* and M^*, they should be understood to be η, α.

3. PRELIMINARIES

Let us denote the intervals $(0, a)$ by I_1, (a, b) by I_2 and in the triple integral equations case (b, ∞) by I_3 but in the quadruple integral equations case (b, c) by I_3 and (c, ∞) by I_4. We define any function $g(x)$ in the whole interval $(0, \infty)$ as

$$g = g_1 + g_2 + g_3 + g_4$$

where $g = g_1$ in I_1 and zero elsewhere with similar definitions for g_2, g_3 and g_4. In case of triple eq. we do not need g_4.

If we write

$$\psi(u) = u^{1+\nu} \phi(u), f(x) = 2^{\mu-2\beta} x^{2\beta-\mu} F(x), g(x) = 2^{-\nu} x^{\nu} G(x)$$

in the sets of eqs (1) and (2), they may be written in the operational form as

$$S \begin{pmatrix} 0, & 0, & k \\ \beta, & \mu - 2\beta, & \beta \end{pmatrix} \{1 + H(x)\} \phi(x) = f(x), \tag{19}$$

$$S \begin{pmatrix} 0, & 0, & k \\ \beta, & \mu - 2\beta, & \beta \end{pmatrix} \{1 + \bar{H}(x)\} \phi(x) = \bar{f}(x), \tag{20}$$

$$S\begin{pmatrix}0, & 0, & 0\\ \nu, & -\nu, & 0\end{pmatrix}\phi(x) = g(x). \tag{21}$$

The eq. (20) is omitted in the case of triple equations.

We regard the function

$$\phi(u) = S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(u), \tag{22}$$

as our trial solution where $h(u)$ is a function as yet undetermined. Substituting this expression for $\phi(u)$ in eqs (19), (20) and (21) and using the formulae (11) and (12), we get

$$\begin{aligned} f(x) &= S\begin{pmatrix}0, & 0, & k\\ \beta, & \mu-2\beta, & \beta\end{pmatrix} S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(u) \\ &\quad + S\begin{pmatrix}0, & 0, & k\\ \beta, & \mu-2\beta, & \beta\end{pmatrix} H(u)\, S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(u), \\ &= T_k(0, \mu-\beta)\, h(x) + E(x), \end{aligned} \tag{23}$$

$$\begin{aligned} \bar{f}(x) &= S\begin{pmatrix}0, & 0, & k\\ \beta, & \mu-2\beta, & \beta\end{pmatrix} S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(u) \\ &\quad + S\begin{pmatrix}0, & 0, & k\\ \beta, & \mu-2\beta, & \beta\end{pmatrix} \bar{H}(u)\, S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(u), \\ &= T_k(0, \mu-\beta)\, h(x) + \bar{E}(x), \end{aligned} \tag{24}$$

$$\begin{aligned} g(x) &= S\begin{pmatrix}0, & 0, & 0\\ \nu, & -\nu, & 0\end{pmatrix} S\begin{pmatrix}k, & 0, & 0\\ 0, & \beta, & \tfrac{1}{2}\beta\end{pmatrix} h(x), \\ &= A_{ik}(\nu, \beta-\nu)\, h(x), \end{aligned} \tag{25}$$

where

$$E(x) = 2^{\mu-\beta} x^{2\beta-\mu} \int_0^\infty u^{1-\beta} h(u)\, du \int_k^\infty w^{1-\mu} H(w) (w^2-k^2)^{\beta/2} \times J_\mu(xw)\, J_\beta\{u\sqrt{(w^2-k^2)}\}\, dw,$$

and $\bar{E}(x)$ is $E(x)$ with bar over H.

Using the formulae (8) and (9) and solving above eqs for $h(x)$ we obtain

$$\begin{aligned} h(x) &= T_{ik}(\mu-\beta, \beta-\mu) f(x) - T_k^{-1}(0, \mu-\beta)\, E(x), && (26) \\ &= T_{ik}(\mu-\beta, \beta-\mu) \bar{f}(x) - T_k^{-1}(0, \mu-\beta)\, \bar{E}(x), && (27) \\ &= A_{ik}(\beta, \nu-\beta)\, g(x). && (28) \end{aligned}$$

4. TRIPLE INTEGRAL EQUATIONS

In this section we are dealing with the set of eq. (1) and we have there $g_3 = 0$, whilst g_1 and f_2 are known functions. Evaluating (28) on I_3 we find (since $g_3 = 0$) that $h_3 = 0$. Hence the upper limit of u in (23) is b. Now evaluating equations (23) on I_1, (26) and (25) on I_2 and (28) on I_1 again, we get the following results:

$$f_1(x) = \binom{x}{o} T_k(0, \mu - \beta) h_1(x) + E(x), \tag{29}$$

$$h_2(u) + \binom{a}{o} T_{ik}(\mu - \beta, \beta - \mu) f_1(x) + \binom{x}{a} T_k^{-1}(0, \mu - \beta) f_2(x) - \binom{x}{o} T_k^{-1}(0, \mu - \beta) E(x), \tag{30}$$

$$g_2(x) = \binom{b}{x} A_{ik}(\nu, \beta - \nu) h_2(x), \tag{31}$$

$$h_1(u) = \binom{b}{a} A_k(\beta, \nu - \beta) g_2(x) + \binom{a}{x} A_k(\beta, \nu - \beta) g_1(x). \tag{32}$$

After eliminating $f_1(x)$ between (29) and (30) and $g_2(x)$ between (31) and (32), we find the functions $h_1(x)$ and $h_2(x)$ satisfy the pair of simultaneous integral equations:

$$h_2(x) = -\begin{pmatrix} x, & a \\ a, & 0 \end{pmatrix} L^*(0, \mu - \beta) h_1(x) + \binom{x}{a} T_k^{-1}(0, \mu - \beta) f_2(x) - \binom{x}{a} T_k^{-1}(0, \mu - \beta) E(x), \tag{33}$$

$$h_1(x) = -\begin{pmatrix} a, & b \\ x, & a \end{pmatrix} M^*(\nu, \beta - \nu) h_2(x) + \binom{a}{x} A_k(\beta, \nu - \beta) g_1(x), \tag{34}$$

where we have used the formulae (17) and (18).

Equations (33) and (34) are two relations between $h_1(x)$ and $h_2(x)$, from which we can determine these unknowns and hence the function ϕ from (22).

5. THE QUADRUPLE INTEGRAL EQUATIONS

Evaluating (24) on I_4, we find (since $g_4 = 0$) that $h_4 = 0$, whilst g_2, f_1 and f_3 are known functions, hence the upper limit of u in (23) and (24) will be c. Hence on evaluating (26) on I_1, (25) on I_2 and (24) on I_3 we arrive at the following results:

$$h_1(x) = \binom{x}{0} T_{ik}(\mu-\beta, \beta-\mu) f_1(x) - \binom{x}{0} T_k^{-1}(0, \mu-\beta) E(x), \tag{35}$$

$$h_2(x) = -\binom{b}{x} A_{ik}^{-1}(\nu, \beta-\nu) \binom{c}{b} A_{ik}(\nu, \beta-\beta) h_3(x)$$

$$\binom{b}{x} A_{ik}^{-1}(\nu, \beta-\nu) g_2(x), \tag{36}$$

$$h_3(x) = \binom{x}{b} T_k^{-1}(0, \mu-\beta) \bar{f}_3(x) - \binom{x}{b} T_k^{-1}(0, \mu-\beta)$$

$$\times \binom{a}{o} T_k(0, \mu-\beta) h_1(x) - \binom{x}{b} T_k^{-1}(0, \mu-\beta)$$

$$\times \binom{a}{o} T_k(0, \mu-\beta) h_2(x) - \binom{c}{o} T_k^{-1}(0, \mu-\beta) \bar{E}(x). \tag{37}$$

Applying the results (13) and (14) to the eqs (36) and (37), we get

$$h_2(x) = \binom{b}{x} A_{ik}^{-1}(\nu, \beta-\nu) g_2(x) - \binom{b,\ c}{x,\ b} M^*(\nu, \beta-\nu) h_3(x), \tag{38}$$

$$h_3(x) = \binom{x}{b} T_k^{-1}(0, \mu-\beta) \bar{f}_3(x) - \binom{x,\ a}{b,\ 0} L^*(0, \mu-\beta) h_1(x)$$

$$-\binom{x,\ a}{b,\ 0} L^*(0, \mu-\beta) h_3(x)$$

$$-\binom{c}{o} T_k^{-1}(0, \mu-\beta) \bar{E}(x). \tag{39}$$

Equations (35), (38) and (39) are three relations between $h_1(x)$, $h_2(x)$ and $h_3(x)$ from which they can be determined and hence eq. (22) will give the unknown function ϕ.

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