An inhomogeneity problem in couple stress theory

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M S received 12 May 1976

ABSTRACT

Using Hilbert theory and Mindlin's couple stress theory, the problem of two-dimensional circular inhomogeneity (when the inserted material is of different size than the size of the cavity and having different elastic constants) is studied in this paper. Stresses could be bounded at infinity. The formulation is valid also for regions other than the circular ones when the elastic constants of inclusion (inserted material) and matrix (outer material) are the same. The problem of circular inclusion when the matrix is finite has also been tackled. Numerical results are in conformity with the fact that the effect of couple stresses is negligible when the ratio of the smallest dimension of the body to the characteristic length is large.

1. INTRODUCTION

The problem of two-dimensional circular inhomogeneity in the infinite medium with couple stresses under uniaxial tension at infinity was solved by Weitsman. The inhomogeneity considered is of the same size as that of the size of the cavity. Weitsman did not give the values of the arbitrary constants involved. The same problem was tackled by Hartmanf and Sih in which the values of the arbitrary constants involved are given.

In both these papers the forms of the functions satisfying the governing differential equations was motivated by Mindlin's paper. This choice of functions does not seem to be appropriate for a general class of problems. The complex variable formulation of the two-dimensional couple stress theory given by Mindlin and Huigol seems to be more appropriate in tackling the class of inclusion problems. This has been done in the present paper with a particular reference to the problem of circular inhomogeneity (when the inserted disc is of bigger size than the size of the cavity) in the infinite medium. The problem has been generalized to the case when the outer boundary of the matrix is a circle.
2. Analysis

Consider an infinite elastic plate in the state of plane strain with a circular hole. The boundary of the hole is given by \(| z | = R (z = x + iy)\) and will be denoted by \(L\). Suppose an elastic solid of dimensions slightly larger than those of the hole but remaining within the limits of proportional elasticity is embedded in the hole. This embedded elastic solid will be called inhomogeneity if its elastic constants are different from those of the outer region and inclusion if the elastic constants are the same as that of the outer region. The outer region will be called matrix.

Let the inhomogeneity in the absence of matrix undergo a prescribed deformation \((\epsilon x, \epsilon y)\) which in the presence of matrix will attain a different equilibrium configuration. If body forces and body couples are zero but couple stresses are taken into account, then the following conditions must hold at the equilibrium boundary \(L\).

\[
\begin{align*}
\nu^+ - \nu^- &= \epsilon x = g_1(t) \tag{1.1} \\
\nu^+ - \nu^- &= \epsilon y = g_2(t) \tag{1.2} \\
\sigma^+_{\phi\phi} + i\sigma^+_{\phi\theta} &= \sigma^-_{\phi\phi} + i\sigma^-_{\phi\theta} \tag{1.3} \\
\mu^+ &= \mu^- \tag{1.4} \\
\omega^+_{\phi\theta} &= \omega^-_{\phi\theta} \tag{1.5}
\end{align*}
\]

where \(t\) is a point on the boundary \(L\); the superscripts + and - stand for the matrix (\(| z | > R\)) and inhomogeneity (\(| z | < R\)) respectively. \(u\) and \(v\) are the displacement components in Cartesian coordinates, \(\tau_{\phi\phi}, \tau_{\phi\theta}\) etc. are the components of asymmetric Cosserat stress tensor in polar coordinates, \(\mu\) is the component of the Cosserat couple-stress tensor in polar coordinates and \(\omega_{\phi\theta}\) is the component of rotation produced by the anti-symmetric part of the shear stresses.

At infinity one of the following conditions may hold

Uniaxial tension at infinity \hspace{1cm} (2.1)
Biaxial tension at infinity \hspace{1cm} (2.2)
Known principal stresses at infinity. \hspace{1cm} (2.3)

The components of Cosserat stress tensor, displacements and couple stresses can be expressed in terms of two complex potentials \(\phi (z)\) and \(\chi (z)\) and a real valued function \(V (z, \bar{z})\). These expressions can be easily derived from Mindlin\(^8\) (equation 31) and Muskhelishvili\(^8\) and are reported by Huigol.\(^5\)

It may be noted that \(U = Re (\bar{z} \phi (z) + \chi (z))\) satisfies the biharmonic equation \(\nabla^4 U = 0\) and the function \(V (z, \bar{z})\) satisfies the differential equation
\[ \nabla^2 V - l^2 \nabla^4 V = 0. \] (3)

The boundary conditions (1.1) to (1.5) in terms of \( \phi (z), \psi (z) = \chi' (z) \) and \( V(z, \tilde{z}) \) can be written as follows

\[
\begin{align*}
\phi^+ (t) - G_1 \psi^+ (t) - G_2 \chi' (t) + 2iG_1 \frac{\partial V^+}{\partial t} \\
= k_1 G_2 \phi^- (t) - G_1 \bar{\psi}^- (t) - G_2 \bar{\chi}^- (t) + 2iG_2 \frac{\partial V^-}{\partial t} \\
+ 2G_1 G_2 (g_1 (t) + ig_2 (t)) \text{ on } L,
\end{align*}
\]

(4.1)

\[
\begin{align*}
\phi^+ (t) + t \phi^- (t) + \psi^+ (t) - 2i \frac{\partial V^+}{\partial t} \\
= \phi^- (t) + t \bar{\phi}^- (t) + \bar{\psi}^+ (t) - 2i \frac{\partial V^-}{\partial t} \text{ on } L
\end{align*}
\]

(4.2)

\[
\text{Re} \left( e^{it} \frac{\partial V^+}{\partial t} \right) = \text{Re} \left( e^{it} \frac{\partial V^-}{\partial t} \right) \text{ on } L
\]

(4.3)

\[ l_1^2 G_1 V^+ = l_2^2 G_2 V^- \text{ on } L. \]

(4.4)

where the elastic constants and characteristic lengths of inhomogeneity and matrix are denoted by the subscripts 1 and 2 respectively; \( k = 3 - 4v \), \( v \) being Poisson ratio, \( G \) is the shear modulus of elasticity and \( l_1 \) and \( l_2 \) are the characteristic lengths of inhomogeneity and matrix respectively.

There is a mistake in Huigol's paper and the equation (1.11) on page 82 should be read as

\[
2G (\mu + iv) = (3 - 4v) \phi (z) - z \bar{\phi}' (z) - \bar{\psi} (z) + 2i \frac{\partial V}{\partial \bar{z}}
\]

(5)

We introduce two new functions \( \theta (z) \) and \( \Omega (z) \) as follows

\[
\begin{align*}
k_2 G_1 \phi^+ (z) + G_2 z \bar{\phi}^- (R^2 / z) + G_2 \bar{\chi}^- (R^2 / z) &= \Omega^+ (z), z \in \text{matrix} \quad (6) \\
k_1 G_2 \phi^- (z) + G_1 z \bar{\phi}^+ (R^2 / z) + G_1 \bar{\chi}^+ (R^2 / z) &= \Omega^- (z), z \in \text{inhomogeneity} \quad (7)
\end{align*}
\]

\[
\begin{align*}
\phi^+ (z) - z \phi^- (R^2 / z) - \phi^- (R^2 / z) &= \theta^+ (z), z \in \text{matrix} \quad (8) \\
\phi^- (z) - z \bar{\phi}^+ (R^2 / z) - \bar{\phi}^+ (R^2 / z) &= \theta^- (z), z \in \text{inhomogeneity} \quad (9)
\end{align*}
\]

It may be noted that \( \Omega^+ (z) \) and \( \theta^+ (z) \) are holomorphic in the region \( |z| > R \) except possibly at infinity and \( \Omega^- (z) \) and \( \theta^- (z) \) are holomorphic in the region \( |z| < R \) except possibly at the origin. Conditions (4.1) and (4.2) can now be rewritten as

\[
\begin{align*}
\Omega^+ (t) - \Omega^- (t) &= 2iG_2 \frac{\partial V^-}{\partial t} - 2iG_1 \frac{\partial V^+}{\partial t} + 2G_1 G_2 \left[ g_1 (t) \\
&+ ig_2 (t) \right] \text{ on } L
\end{align*}
\]

(10.1)
The solutions of Hilbert problems (10.1) and (10.2) can be written if the function $V$ satisfying (3) can be found.

The solutions of (3) can be written as follows:

$$V^+ = \sum_{n=0}^{\infty} \left( \frac{b_n}{z^n} + \frac{\delta_n}{z^n} \right) + \sum_{n=1}^{\infty} K_n \left( \frac{r}{l_2} \right) \{ c_n \sin n\theta + d_n \cos n\theta \}, \quad r > R \quad (11)$$

$$V^- = \sum_{n=0}^{\infty} (a_n z^n + \alpha_n z^n) + \sum_{n=1}^{\infty} I_n \left( \frac{r}{l_2} \right)$$

$$+ \{ c'_n \sin n\theta + d'_n \cos n\theta \}, \quad r < R. \quad (12)$$

$K_n$ and $I_n$ are Bessel functions of the second kind and order $n$; $c_n$, $d_n$, $c'_n$ and $d'_n$ are real constants which together with $a_n$ and $b_n$ are to be determined. The requirement of the periodicity of $V$ in $\theta$ imposes the condition that the summation in the second series of (11) and (12) should be over positive integral values.

Using (11) and (12) various quantities in the right hand side of (10.1) and (10.2) can be calculated and $\Omega(\omega)$ and $\theta(\omega)$ can be found using Hilbert theorem. $\phi^+(\omega)$ and $\phi^-(\omega)$ can be calculated from the following relations:

$$\phi^+(\omega) = \frac{G_2 \phi^+(\omega) + \Omega^+(\omega)}{G_2 + G_1 k_2}$$

and

$$\phi^-(\omega) = \frac{G_1 \phi^-(\omega) + \Omega^-(\omega)}{G_1 + G_2 k_1}$$

and are given by

$$\phi^+(\omega) = \frac{i(G_2 - G_1)}{2(G_2 + G_1 k_2)} \left[ \frac{1}{l_2} \sum_{n=2}^{\infty} K'_n \left( \frac{R}{l_2} \right) (d_n + i c_n) \frac{R^{n-1}}{z^{n-1}} \right.$$

$$+ \sum_{n=2}^{\infty} nK_n \left( \frac{R}{l_2} \right) (d_n + i c_n) \frac{R^{n-2}}{z^{n-2}} \left. + \frac{G_2 G_1 (\epsilon_1 - \epsilon_2) R^2}{(G_2 + G_1 k_2)z} \right]$$

$$+ \left[ \frac{G_2 A_1 + C_1}{G_2 + G_1 k_2} \right] \frac{G_1 - G_2}{G_2 + G_1 k_2} \frac{A_2}{z} \quad (13)$$
\[ \phi^-(z) = \frac{i (G_1 - G_2)}{2 (G_1 + G_2 k_1)} \left[ \sum_{n=1}^{\infty} I'_n \left( \frac{R}{l_1} \right) (d'_n - i c'_n) \frac{z^{n+1}}{R^{n+1}} \right] \]
\[ - \sum_{n=1}^{\infty} n I'_n \left( \frac{R}{l_2} \right) (d'_n - i c'_n) \frac{z^{n+1}}{R^{n+2}} \right] - \frac{G_1 G_3 (\epsilon_1 + \epsilon_2)}{G_1 + G_2 k_1} z
\]
\[ + \frac{(G_1 A_1 + C_1)}{G_1 + G_2 k_1} z. \] (14)

The constants \( A_1, C_1 \) and \( A_2 \) are to be determined from the conditions at infinity and the conditions that stresses and displacements should remain bounded at the origin.

For the uniaxial tension \( p \) in the \( y \) direction
\[ A_1 \times 8 \{ G_1 + G_2 (1 - 2\nu_y) \} = [4G_1G_2 (\epsilon_1 + \epsilon_2) - p \{ G_1 (1 - \nu_y) \}
\[ - G_2 (1 - \nu_y) \}] \] (15)
\[ G_2 A_1 + C_1 = p/8 (G_2 + G_1 k_2) \] (16)
\[ A_2 = pR^2/4. \]

For the biaxial tensions \( q \) and \( p \) in the \( x \) and \( y \) directions respectively
\[ 4A_2 = (p - q) R^2 \] and \( A_1 \) and \( C_1 \) are given by (15) and (16) with \( p \) replaced by \( p + q \).

When the principal stresses \( N_1 \) and \( N_2 \) act at infinity and the angle between \( N_1 \) and the \( x \)-axis is \( \delta \) then
\[ A_2 = - \frac{(N - N_2) e^{2\alpha}}{4} \]
and \( A_1 \) and \( C_2 \) are given by (15) and (16) with \( p \) replaced by \( N_1 + N_2 \).

Using \( V \) given in (11) and (12) in (4.3) and (4.4), we get
\[ l_1 R^{n+1} K'_n \left( \frac{R}{l_2} \right) (d_n + ic_n) - l_2 R^{n+1} I'_n \left( \frac{R}{l_1} \right) (d'_n + ic'_n) \]
\[ - 2l_1 l_2 n b_n - 2l_1 l_2 n a_n R^{2n} = 0, \quad n \geq 1 \] (17)
\[ l_1^2 G_1 R^n K_n \left( \frac{R}{l_2} \right) (d_n + ic_n) - l_2^2 G_2 R^n I_n \left( \frac{R}{l_1} \right) (d'_n + ic'_n) \]
\[ + 2l_1^2 G_2 b_n - 2l_2^2 G_2 a_n R^{2n} = 0, \quad n \geq 1 \] (18)
\[ l_1^2 G_1 \Re (b_0) = l_2^2 G_2 \Re (a_0). \]

The condition
\[ \text{Im} [\phi^-(z)] \]
provides the remaining equations to determine unknowns completely. Im stands for the imaginary part of a complex quantity.

\[
b_n = -2 (1 - v_2) l_2 \frac{(G_2 - G_1)}{G_2 + G_1 k_2} (1 - n) R^{n-1} K_{n-1} \left( \frac{R}{l_2} \right) (d_n + i c_n), \quad n \geq 3. \tag{19}
\]

\[
\text{Re} (b_0) = 8 (1 - v_2) l_2^2 \text{Im} \left( \frac{G_2 A_1 + C_1}{G_2 + G_1 k_2} \right), \quad b_1 = 0 \tag{20}
\]

\[
a_n = 2 (1 - v_3) \frac{G_1 - G_2}{(G_1 + G_2 k_1)} \frac{(1 + n) l_1^2}{R^{n+2}} \left[ \frac{R}{l_1} I_{n-1} \left( \frac{R}{l_1} \right) - 2n I_n \left( \frac{R}{l_1} \right) \right]
\]

\[
(d' n - i c' n), \quad n \geq 1 \tag{21}
\]

\[
b_2 = 2 (1 - v_2) R l_2 \frac{(G_2 - G_1)}{G_2 + G_1 k_2} K_1 \left( \frac{R}{l_2} \right) (d_2 + i c_2)
\]

\[
+ \frac{4i (1 - v_2)}{G_2 + G_1 k_2} G_1 G_2 l_2^2 R^2 (\epsilon_1 - \epsilon_2)
\]

\[
- \frac{4i (1 - v_2) l_2^2 (G_2 - G_1) A_2}{G_2 + G_1 k_2}. \tag{22}
\]

It can be shown that \(a_2, b_2, c_2, c', a_0\) and \(b_0\) are the only non-zero constants.

\[
c_2 = \frac{1}{\Delta} \left\{ h_3 + 2 l_2^2 h_2 \text{Re} (A_2) \right\} \left\{ \{R^3 l_2 + 3 R l_1^2 l_2 (1 + g)\} I_1 (\lambda_2) + \{2 R^2 (g - 1) l_1 l_2 - 12 h_1 l_1^3 l_2 (1 + g)\} I_2 (\lambda_1) \right\} \tag{23}
\]

\[
c'_2 = -\frac{R^3 l_2 K_1 (\lambda_2)}{\Delta} \left\{ 2 l_2^2 \text{Re} (A_2) h_2 + h_3 \right\} \tag{24}
\]

\[
a_2 = -\frac{3i h_1 l_1^2}{4 R^4} \left\{ \lambda_1 I_1 (\lambda_1) - 4 I_2 (\lambda_2) \right\} c'_2 \tag{25}
\]

\[
b_2 = -\frac{i R}{4} h_2 l_2 K_1 (\lambda_2) c_2 + \frac{h_3}{4} - \frac{i}{2} l_2^2 A_2 \tag{26}
\]

\[
d_2 = -\frac{2 l_2^2 h_2 \text{Im} (A_2)}{\Delta} \left\{ \{R^3 l_2 + 3 R l_1^2 l_2 (1 + g)\} \times I_1 (\lambda_1) + \{2 R^2 (g - 1) l_1 l_2 - 12 h_1 l_1^3 l_2 (1 + g)\} I_2 (\lambda_1) \right\} \tag{27}
\]

\[
d'_2 = -\frac{2 R^3 l_1 l_2^2 h_2 \text{Im} (A_2)}{\Delta} K_1 (\lambda_2), \tag{28}
\]

\[
h_1 = 8 (1 - v_1) \frac{(G_1 - G_2)}{G_1 + G_2 k_1}.
\]
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\[ h_2 = 8 \left( 1 - v^2 \right) \frac{(G_1 - G_2)}{G_2 + G_1 k_2}, \]

\[ h_3 = \frac{16 \left( 1 - v^2 \right) l_1^2 G_1 G_2 R^2 (\epsilon_1 - \epsilon_2)}{G_2 + G_1 k_2}, \]

\[ g = \frac{l_1^2 G_2}{l_1^2 G_1}, \quad \lambda_1 = \frac{R}{l_1}, \quad \lambda_2 = \frac{R}{l_2} \]

and

\[
\triangle = \left\{-3 h_1 R^4 g l_1^2 + h_2 l_2^2 R^4 + 3 h_1 h_2 R^2 l_1^2 l_2^2 (1 + g) \right\}
\times K_1 (\lambda_2) I_1 (\lambda_1) - 2K_2 (\lambda_2) I_2 (\lambda_1) R^2 l_1 l_2 (2R^2 (g - 1) - 12 l_1^2 (1 + g))
- 2R^2 K_2 (\lambda_2) I_1 (\lambda_1) \left\{ R^3 l_2 + 3R l_1^2 l_2 l_1 (1 + g) \right\}
+ K_1 (\lambda_2) I_2 (\lambda_1) \left\{ -2 R^5 g l_1 + 12 h_1 R^3 l_1^3 g \right\}
+ 2 h_2 l_1 l_2^2 R^3 (g - 1) - 12 h_1 h_2 R l_2 l_1^3 (1 + g) \right\}
\] (29)

when \( \epsilon_1 = \epsilon_2 = 0 \) and only uniaxial tension in the y direction is considered, the values of the constants \( a_2, b_2, c_2, c'_2 \) agree with the values of the corresponding constants in [2].

\( \psi^+ (z) \) and \( \psi^- (z) \) can be determined from (8) and (9). Finally,

\[ U^+ = \text{Re} \{ \hat{\psi}^+ (z) + \chi^+ (z) \} \]
\[ = [ G_2 (1 + k_2) R^2 c'_2 \{ \lambda_1 I_1 (\lambda_1) - 4I_2 (\lambda_1) \} ] + G_1 (1 + k_2) a R^2 c_2 \{ \lambda_2 K_1 (\lambda_2) + 2K_2 (\lambda_2) \}
+ 2a \{ 2G_2 + G_1 (k_2 - 1) \} R^2 K_2 (\lambda_2) c_2 + 8 (G_1 + G_2 k_2) \text{Im} (b_2)
- 2R^4 a G_1 G_2 (\epsilon_1 - \epsilon_2) + 2R^2 (G_2 - G_1) a \text{Re} (A_2) \times
\times \frac{\cos 2\theta}{4 (G_1 + G_2 k_2) R^2} + \left\{ G_2 - G_1 \right\} \lambda_2 + 2G_1 G_2 (\epsilon_1 - \epsilon_2) R^2
+ 2 (G_1 - G_2) \text{Re} (A_2) \frac{\cos 2\theta}{2 (G_2 + G_2 k_2)}
+ \text{Re} (A_2) \frac{r^2}{2R^2} \cos 2\theta + \frac{(G_2 A_1 + C_1) r^2}{2G_2 + G_1 k_2} - 2A_1 k^2 \log r
+ \left\{ 1 - \frac{R^2}{2r^2} \right\} \frac{(G_1 - G_2)}{G_2 + G_2 k_2} + \frac{r^2}{2R^2} \sin 2\theta \text{Im} (A_2) \right\} \] (30)

and

\[ U^- = \text{Re} \{ \hat{\psi}^- (z) + \chi^- (z) \} \]
\[ = [- G_1 (1 + k_2) a\lambda_2 c_2 K_1 (\lambda_2) + 8 (G_1 + G_2 k_2) R^2 \text{Im} (A_2) \]
where

\[ \alpha = \frac{G_1 + G_2 k_1}{G_2 + G_1 k_2}. \]

When \( \epsilon_1 = \epsilon_2 = 0 \) and only uniaxial tension in the \( y \) direction is considered, \( U^+ \) and \( U^- \) agree totally with the results of Hartranft and Sih. 2

The results for circular inclusion in infinite region are well known. 7 If couple stress effects are also considered in the problem of circular inclusion in infinite region (with stresses vanishing at infinity) then the results can be easily obtained from \( \phi(z) \), \( \psi(z) \) and \( V \) obtained above. \( \tau_{rr} \), \( \tau_{r\theta} \) and \( \tau_{\theta\theta} \) with and without couple stress effects have been calculated at the equilibrium boundary for circular inclusion in an infinite region and are reported in table 1.

It may be noted that the definitions of \( \Omega(z) \) and \( \theta(z) \) as given in (6)–(9) hold only for circular boundaries and so if the boundary \( L \) is other than a circle, the equations (10.1) and (10.2) are no more valid. However if the elastic constants of inclusion and matrix are the same then one can proceed directly with the equations (4.1) to (4.4) and obtain the required functions \( \phi(z) \), \( \psi(z) \) and \( V(z, \bar{z}) \).

Next, we consider a two-dimensional plane strain problem of circular inclusion in a finite matrix. The outer boundary of matrix is taken as a circle of radius \( r' \) and will be denoted by \( L_0 \). The problem of circular inhomogeneity in a finite matrix is still an open problem. If \( \phi_0(z) \), \( \psi_0(z) \), and \( V_0(z, \bar{z}) \) are the new functions in terms of which Cosserat stress tensor, displacements etc. can be written then

\[
\phi_0(z) = \frac{G (\epsilon_1 - \epsilon_2) R^2}{(1 + k) z} + \sum_{n=0}^{\infty} A_n z^n, \quad |z| > R
\]
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\[ \psi_0(z) = -\frac{R^2}{2\pi z} c_{02} \left\{ \lambda_0 I_1(\lambda_0) - 4I_2(\lambda_0) \right\} - \frac{c_{02} R^2}{2\pi z} \]

\[ \times \left\{ \lambda_0 K_1(\lambda_0) + 2K_2(\lambda_0) \right\} + \frac{4i b_{02}}{z^3} + \frac{R^4}{z^4} \frac{G(\epsilon_1 - \epsilon_2)}{(1 + k)} \]

\[ = -\frac{2GR^2(\epsilon_1 - \epsilon_2)}{(k + 1)z} + \sum_{n=0}^{\infty} B_n z^n, \quad |z| > R \]

\[ = -\frac{\lambda_0}{2} K_1(\lambda_0) c_{02} \frac{z}{R^2} - \frac{\lambda_0 z}{2R^2} I_1(\lambda_0) \]

\[ + \frac{G(\epsilon_1 - \epsilon_2)}{(1 + k)z} + \sum_{n=0}^{\infty} B_n z^n, \quad |z| < R. \] (33)

\[ V_0(z, \bar{z}) = \frac{b_{02}}{z^2} + \frac{\bar{b}_{02}}{\bar{z}^2} + c_{02} K_2 \left( \frac{r}{l} \right) \sin 2\theta \]

\[ = \sum_{n=0}^{\infty} \left\{ \tilde{a}'_n z^n + \bar{a}'_n \bar{z}^n \right\} + \sum_{n=1}^{\infty} I_n \left( \frac{r}{l} \right) \left\{ c''_n \sin n\theta \right\} + \tilde{d}'' \cos n\theta, \quad |z| > R \] (34)

\[ = \sum_{n=1}^{\infty} \left( a'_n z^n + \bar{a}'_n \bar{z}^n \right) + I_2 \left( \frac{r}{l} \right) c'_{02} \sin 2\theta \]

\[ + \sum_{n=1}^{\infty} I_n \left( \frac{r}{l} \right) \left\{ c''_n \sin n\theta + d''_n \cos n\theta \right\}, \quad |z| < R \]

where \( \lambda_0 = \frac{R}{l} \), \( l \) is the characteristic length of the inclusion as well as matrix,

\[ c_{02} = -\frac{8(1 - \nu) G l^2 \lambda_0 (\epsilon_1 - \epsilon_2) I_1(\lambda_0)}{(1 + k)} \]

\[ c'_{02} = \frac{8(1 - \nu) G l^2 \lambda_0 (\epsilon_1 - \epsilon_2) K_1(\lambda_0)}{(1 + k)} \] (35)

\[ b_{02} = \frac{4i(1 - \nu) G l^2 R^2 (\epsilon_1 - \epsilon_2)}{(1 + k)} \]

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Table I. \(\tau_{rr}, \tau_{\theta \theta}\) and \(\tau_{\theta r}\) at the equilibrium boundary for circular inclusion in an infinite region. The first value of these stresses in the table refers to couple stress effects and the second value refers to without couple stresses.

\[
\epsilon_1 = -\epsilon_2, \quad \nu = 0.25, \quad P = (1 + k)/2G\epsilon_1
\]

<table>
<thead>
<tr>
<th>(R/I)</th>
<th>(\theta)</th>
<th>(\tau_{rr}/P)</th>
<th>(\tau_{\theta r}/P)</th>
<th>(\tau_{\theta \theta}/P) matrix</th>
<th>(\tau_{\theta \theta}/P) inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 (-2.1410)</td>
<td>0 (-1.0000)</td>
<td>0 (0.0000)</td>
<td>-1.8589</td>
<td>-3.0000</td>
</tr>
<tr>
<td>(\pi/6)</td>
<td>0 (-1.0705)</td>
<td>0 (-0.5000)</td>
<td>1.6453</td>
<td>0.8660</td>
<td>-0.9294</td>
</tr>
<tr>
<td>1.0</td>
<td>(\pi/3)</td>
<td>0 (1.0705)</td>
<td>0 (0.5000)</td>
<td>0.8660</td>
<td>1.6453</td>
</tr>
<tr>
<td>(\pi/2)</td>
<td>0 (2.1410)</td>
<td>0 (0.0000)</td>
<td>0 (0.0000)</td>
<td>1.8589</td>
<td>3.0000</td>
</tr>
<tr>
<td>2(\pi/3)</td>
<td>0 (1.0705)</td>
<td>0 (-0.5000)</td>
<td>-1.6453</td>
<td>-0.8660</td>
<td>0.9294</td>
</tr>
<tr>
<td>5(\pi/6)</td>
<td>0 (-1.0705)</td>
<td>0 (-0.5000)</td>
<td>-1.6453</td>
<td>-0.8660</td>
<td>-0.9294</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0 (-2.1410)</td>
<td>0 (-1.0000)</td>
<td>0 (0.0000)</td>
<td>-1.8589</td>
<td>-3.0000</td>
</tr>
<tr>
<td>5.0</td>
<td>0 (-1.6713)</td>
<td>0 (1.0000)</td>
<td>0 (0.0000)</td>
<td>-2.328</td>
<td>-3.0000</td>
</tr>
<tr>
<td>(\pi/6)</td>
<td>0 (-0.8356)</td>
<td>0 (-0.5000)</td>
<td>0.7960</td>
<td>0.8660</td>
<td>1.1643</td>
</tr>
<tr>
<td>25.0</td>
<td>0 (-1.2127)</td>
<td>0 (1.0000)</td>
<td>0 (0.0000)</td>
<td>-2.7872</td>
<td>-3.0000</td>
</tr>
<tr>
<td>(\pi/6)</td>
<td>0 (-0.6063)</td>
<td>0 (-0.5000)</td>
<td>0.7856</td>
<td>0.8660</td>
<td>1.3936</td>
</tr>
</tbody>
</table>

For other values of \(\theta\) (\(\pi/3, \pi/2\) etc.) the pattern of values of stresses for \(R/I = 5.0\) and \(R/I = 25.0\) is the same as given in the case \(R/I = 1.0\).

\(c^r, d^r\) are real constants which along with \(A_n, B_n\) and \(a'_n\) are to be determined.

On the outer boundary \(L_0\)

\[
\phi_0 (t) + i \phi_0' (t) + \psi_0 (t) - 2i \frac{\partial V_0}{\partial \tau} = 0
\]

and

\[
\mu_r = \text{Re} \left( e^{ib} \frac{\partial V_0}{\partial \tau} \right) = 0.
\]

Further

\[
V_0 - l^2 \nabla^2 V_0 = 8 (1 - \nu) l^2 \text{Im} \{\phi_0' (z)\}.
\]

The conditions (36)-(38) are sufficient to determine the unknowns. If the elastic constants of inclusion and matrix are different then the expression for \(V_0\) as given in (34) will not satisfy (38) and so the problem of inhomogeneity
in a finite matrix is still an open problem. The constants $c''_n$, $d''_n$ etc. are given below

$$c''_2 = \frac{24}{\Delta_1} \frac{(1 - \nu)}{R^2} \left[ - \frac{2G(\epsilon_1 - \epsilon_2)R^2}{(k + 1)} \left( 1 - \frac{R^2}{r''_l} \right) - c''_{o2} \frac{R^2}{r''_l} \left( \lambda_0 I_1 (\lambda_0) - 4I_2 (\lambda_0) \right) - c''_{o2} \frac{R^2}{r''_l} \left[ \lambda_0 K_1 (\lambda_0) + 4 K_2 (\lambda_0) \right] + c_{o2} \left( \lambda' K_1 (\lambda') + 4K_2 (\lambda') \right) \right] + 4i\delta_{o2} \frac{l}{\Delta_1}$$

$$- \frac{c_{o2}}{\Delta_1} r''_l K''_2 (\lambda')$$

$$a'_{o} = - 4i (1 - \nu) l^2 A_1, \quad A_1 = \frac{G R^2 (\epsilon_1 + \epsilon_2)}{(k + 1) r''_l}$$

$$48 (1 - \nu) l^2 r'^4 A_3 = 4i\delta_{o2} + c_{o2} r''^2 \left[ \lambda' K_1 (\lambda') + 2K_2 (\lambda') \right]$$

$$- c''_2 r''_l \left( \lambda' I_1 (\lambda') - 2I_2 (\lambda') \right)$$

$$a''_2 = - 12i (1 - \nu) l^2 A_3$$

$$B_1 = - \frac{G}{(1 + k)} (\epsilon_1 - \epsilon_2) \frac{R^2}{r''_l} + c_{o2} K_1 (\lambda')/2r' l - 4ia''_2$$

$$- 3A_3r''_l - c''_2 I_1 (\lambda')/2r' l$$

where

$$\lambda' = \frac{r'}{l}$$

and

$$\Delta_1 = \left\{ 24 (1 - \nu) + \lambda'^2 \right\} r' l^2 I_1 (\lambda')$$

$$- 2 \left\{ 48 (1 - \nu) + \lambda'^2 \right\} l^3 I_2 (\lambda').$$

The remaining constants are all zero. The non-zero constants given by (39)-(43) when substituted in (32), (33) and (34) will determine $\phi_0$, $\psi_0$ and $V_0 (z, \bar{z})$. When the couple stress effects are not considered, they reduce to the corresponding functions. Then $\tau_{rr}$, $\tau_{r\theta}$ and $\tau_{\theta\theta}$ can be calculated from these functions with the help of following formulae:

$$\tau_{rr} + \tau_{\theta\theta} = 4 \text{Re} \left\{ \phi' (z) \right\}$$

$$\tau_{rr} + i\tau_{r\theta} = \phi' (z) + \overline{\phi' (z)} - \bar{z} \frac{\partial}{\partial z} \psi' (z) - \frac{\partial}{\partial \bar{z}} \psi' (z)$$

$$- 2i \frac{\partial^2 V}{\partial z \partial \bar{z}} + 2i \frac{\partial^2 V}{\partial z^2 \partial \bar{z}}$$

These stresses at the equilibrium boundary $|z| = R$ are given as follows

$$\tau_{rr} = - [B_1 + 48 (1 - \nu) l^2 A_3 + 2c''_2 \left\{ \lambda_0 I_1 (\lambda_0) - 3I_2 (\lambda_0) \right\}/R^2]$$
\[ x \cos 2\theta + \left[ C_{\theta 2}K_1(\lambda_0)/2RI - 3e_{02} \lambda_0 I_1(\lambda_0) - 4I_\theta(\lambda_0)\right]/2R^2. \]
\[ - G (\epsilon_1 - \epsilon_2)/(1 + k) \cos 2\theta - 2G (\epsilon_1 + \epsilon_2)/(1 + k), \] (47)
\[ \tau_{\theta\theta} = \left[ B_1 + 48 (1 - \nu) \frac{A_3}{I} - c^2 \left\{ \lambda_0 I_1(\lambda_0) - 6I_\theta(\lambda_0) \right\}/ \right. 
\[ R^2 + 6A_3 R^2 \cos 2\theta - \left[ C_{\theta 2} K_1(\lambda_0)/2RI + 3e_{02} \lambda_0 I_1(\lambda_0) 
\[ - 4I_\theta(\lambda_0)/2R^2 - G(\epsilon_1 - \epsilon_2)/(1 + k) \right] \sin 2\theta \] (48)
\[ \tau_{\theta\theta} = 4A_1 + 4 \frac{3A_3 R^2 - G(\epsilon_1 - \epsilon_2)/(1 + k)}{ \cos 2\theta - \tau_{rr} \] (49)
\[ \tau_{\theta\theta} = 4A_1 + 12 A_3 R^2 \cos 2\theta - 4G(\epsilon_1 + \epsilon_2)/(1 + k) - \tau_{rr}. \] (50)

3. DISCUSSION

By suitably choosing the values of the constants \( A_1, C_1 \) and \( A_2 \) in (13) and (14), the problem of circular inhomogeneity in an infinite medium subjected to all round uniform tension at infinity or subjected to pure shear at infinity can be tackled. If \( \epsilon_1 = \epsilon_3 \), stresses vanish at infinity and elastic constants of inclusion and matrix are the same then the stresses are independent of couple stress effects when circular inclusion in an infinite medium is considered or circular inclusion in a circular region is considered. This was expected as no rotation gradients are induced.

Table 2. \( \tau_{rr}, \tau_{r\theta} \) and \( \tau_{\theta\theta} \) at the equilibrium boundary for circular inclusion in a circular region.

The first value of these stresses in the table refers to couple stress effects and the second value refers to without couple stresses.

| \( R/I \) | \( \theta \) | \( \epsilon_1 = - \epsilon_3 \), \( \nu = 0.25 \), \( P = (1 + k)/2G\epsilon_1 \), \( r'/k = 10.0 \) |
|--------|--------|-------------------|----------------|-------------------|-------------------|
| 0      | -1.2964 | -0.9603 | 0.0000 | 0.0000 | -2.7010 | -3.0785 | 1.2986 | 0.9614 |
| \( \pi/6 \) | -0.6482 | -0.4801 | 1.0601 | 0.8321 | -1.3506 | -1.5342 | 0.6493 | 0.4807 |
| 1.0    | \( \pi/3 \) | 0.6482 | 0.4801 | 1.0601 | 0.8321 | 1.3506 | 1.5342 | -0.6493 | -0.4807 |
| 1/2    | 1.2964 | 0.9603 | 0.0000 | 0.0000 | 2.7010 | 3.0785 | -1.2986 | -0.9614 |
| 2\( \pi/3 \) | 0.6482 | 0.4801 | -1.0601 | -0.8321 | 1.3506 | 1.5342 | 0.6493 | 0.4807 |
| 5\( \pi/6 \) | -0.6482 | -0.4801 | 1.0601 | 0.8321 | -1.3506 | -1.5342 | 0.6493 | 0.4807 |
| \( \pi \) | -1.2964 | -0.9603 | 0.0000 | 0.0000 | -2.7010 | -3.0785 | 1.2986 | 0.9614 |
| 5.0    | 0.0000 | -0.9603 | 0.0000 | 0.0000 | -3.0052 | -3.0785 | 0.9947 | 0.9614 |
| 5\( \pi/6 \) | -0.4967 | -0.8276 | 0.6481 | 0.8321 | -1.5026 | -1.5342 | 0.4973 | 0.4807 |
| 25.0   | 0.0000 | -0.9623 | 0.0000 | 0.0000 | -3.0384 | -3.0785 | 0.9635 | 0.9614 |
| 5\( \pi/6 \) | -0.5000 | -0.8276 | 0.6481 | 0.8321 | -1.5182 | -1.5342 | 0.4817 | 0.4807 |

For other values of \( \theta \) (\( \pi/3 \), \( \pi/2 \) etc.) the pattern of values of stresses for \( R/I = 5.0 \) and \( R/I = 25.0 \) is the same as given in the case \( R/I = 1.0 \).
Several interesting results can be obtained as particular cases of the results given above and some of them have already been mentioned earlier. By taking $G_2/G_1 = \infty$ and considering appropriate stresses at infinity, the results arrived at by Mindlin can be obtained. The results for rigid inclusion can be obtained by taking $G_2/G_1 = 0$.

$\tau_{rr}$, $\tau_{\theta\theta}$ and $\tau_{r\theta}$ have been calculated at the equilibrium boundary and are reported in tables 1 and 2. The tables are self explanatory. It can be easily seen from the tables that couple stress effects are small if the ratios $R/l$ is $25 \cdot 0$ and quite appreciable if $R/l = 1 \cdot 0$. It is well known that if in the problem of circular inclusion in a circular region (with no couple stress effects) the ratio of inner radius of the circle to the outer radius is small then the results for circular inclusion in an infinite medium can be obtained approximately. Comparison of the two tables confirms in this fact.

REFERENCES