Poisson–Jacobi transforms

R. S. PATHAK AND H. D. CHAUBEY

Department of Mathematics, Banaras Hindu University, Varanasi 221005

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Abstract

The solutions of the Jacobi difference heat equation \( \nabla_n v(n, t) = (\partial^2 / \partial t) v(n, t) \) are considered. The theory of Poisson–Jacobi transform is developed to solve the equation. Convergence and inversion theorems are established.

1. Introduction

Scott and Debnath obtained the solution of the Jacobi differential heat equation

\[
[aD_x (1 - x^2) D_x + (\mu x + \nu) D_x] v(x, t) = CD_t v(x, t) - 1 < x < 1, \quad 0 < t < \infty
\]

by means of the Jacobi transform. They stated various properties of the solutions. In this paper we study solutions of the Jacobi difference heat equation

\[
\nabla_n v(n; t) = \frac{\partial}{\partial t} v(n; t), \quad n = 0, 1, 2, \ldots, \quad 0 < t \leq \infty
\]

where \( \nabla_n \) is the difference operator defined by

\[
\nabla_n f(n) = \frac{1}{A_n} [f(n + 1) - B_n f(n) + C_n f(n - 1)],
\]

where

\[
2(n + 1)(n + a + \beta + 1) A_n = (2n + a + \beta + 1)(2n + a + \beta + 2)
\]

\[
2(n + 1)(n + a + \beta + 1)(2n + a + \beta) B_n = (a^2 - \beta^2)(2n + a + \beta + 1)
\]
Poisson–Jacobi transform is introduced which is useful to solve the difference equation. Asymptotic properties, convergence and inversion theorems are established. The results are analogous to those of Cholewinski and Haimo on Laguerre difference heat equation.

2. Definitions and Preliminary Results

Let \( P_n^{(\alpha, \beta)}(x) \) be the Jacobi polynomial in the usual normalization, that is if \( \alpha > -1, \beta > -1, \)

\[
2^n n! \ P_n^{(\alpha, \beta)}(x) = (-1)^n (1 - x)^{-\alpha} (1 + x)^{-\beta} \left( \frac{d}{dx} \right)^n [(1 - x)^{\alpha+n} (1 + x)^{\beta+n}] \tag{2.1}
\]

These polynomials satisfy the orthogonality relations

\[
\int_{-1}^{1} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) d\Omega_{\alpha, \beta}(x) = \delta(n, m) h_{\alpha, \beta}(n) \tag{2.2}
\]

where

\[
h_{\alpha, \beta}(n) = \frac{(2n + \alpha + \beta + 1) n! (n + \alpha + \beta + 1)}{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \tag{2.3}
\]

and

\[
d\Omega_{\alpha, \beta}(x) = (1 - x)^{\alpha} (1 + x)^{\beta} dx \tag{2.4}
\]

The Jacobi polynomial satisfies the difference equation

\[
\nabla P_n^{(\alpha, \beta)}(x) = x P_n^{(\alpha, \beta)}(x) \quad n = 0, 1, \ldots \tag{2.5}
\]

where

\( P_{-1}^{(\alpha, \beta)}(x) \) is interpreted as zero.

Let \( f(n) \) be a real function defined for \( n = 0, 1, 2, \ldots \), then Jacobi transform of \( f(n) \) is defined by

\[
f(x) = \sum_{n=0}^{\infty} f(n) P_n^{(\alpha, \beta)}(x) h_{\alpha, \beta}(n). \tag{2.6}
\]
The inversion gives

\[ f(n) = \int_{-1}^{1} \hat{f}(x) \, P_n^{(\alpha, \beta)}(x) \, d\Omega_{\alpha, \beta}(x). \] (2.7)

Using (2.5) we find that

\[ (\nabla_n \hat{f})(x) = x \hat{f}(x). \] (2.8)

More generally, if \( p(x) \) is an arbitrary polynomial, then

\[ [p \nabla_n \hat{f}](x) = p(x) \hat{f}(x) \] (2.9)

and

\[ [p \nabla_n f](n) = \int_{-1}^{1} \hat{f}(x) \, p(x) \, P_n^{(\alpha, \beta)}(x) \, d\Omega_{\alpha, \beta}(x). \] (2.10)

**Definition 2.1**

The Jacobi difference heat equation is given by

\[ \nabla_n v(n, t) = (\frac{\delta}{\delta t}) v(n, t). \] (2.11)

Its fundamental solution is the function \( h(n, t) \) given by

\[ h(n, t) = \int_{-1}^{1} e^{tx} \, P_n^{(\alpha, \beta)}(x) \, d\Omega_{\alpha, \beta}(x) \]

\[ = e^{-t} \int_{-1}^{1} e^{(1+x)} \, P_n^{(\alpha, \beta)}(x) \, (1-x)^{\alpha} \, (1+x)^{\beta} \, dx. \] (2.12)

Now evaluating the integral on the right by means of the formula (2, p. 284 (1)) we have

\[ h(n, t) = e^{t} \int_{-1}^{1} e^{2(\alpha+\beta+1+n)} \, \frac{\Gamma(\alpha + n + 1) \, \Gamma(1 + \beta + n)}{n! \, \Gamma(\alpha+\beta+2n+2)} \times \]

\[ {}_{1}F_{1} \left( \begin{array}{c} 1 + \alpha + n \\ \alpha + \beta + 2n + 2 \end{array} ; -2t \right) \]

\[ (\alpha > -1, \quad \beta > -1). \] (2.13)

Corresponding to \( h(n, t) \), we define its conjugate \( h(n^*, t) \) by

\[ h(n^*, t) = \int_{-1}^{1} e^{tx} \, P_n^{(\alpha, \beta)}(-x) \, d\Omega_{\alpha, \beta}(x). \] (2.14)
Changing the variable $x$ to $-x$ in the integral (2.14) and then evaluating it by means of (2, p. 284 (3)) we have

$$h(n^*, t) = e^t \sum_{r=0}^{\infty} \frac{(-2t)^r}{r!} \frac{2^{a+\beta+1} \Gamma(a+1+r) \Gamma(\beta+1)}{\Gamma(a+\beta+2+r)} \times$$

$$\phantom{h(n^*, t)} {}_2F_1\left(-n, a + \beta + n + 1, \beta + 1; 1\right)$$

$$(a > -1, \beta > -1)$$

In addition, we introduce associated functions $b(n, m, k)$ and $b(n^*, m, k)$. We define

$$b(n, m, k) = \int_{-1}^{1} P_n^{(a, \beta)}(x) P_m^{(\alpha, \beta)}(x) P_k^{(\alpha, \beta)}(x) d\Omega_{\alpha, \beta}(x), \quad (2.16)$$

By inversion formula for the Jacobi transform, we find

$$b(n, m, k)(x) = \sum_{k=0}^{\infty} b(n, m, k) P_n^{(a, \beta)}(x) h_{a, \beta}(k) = P_n^{(a, \beta)}(x) P_m^{(a, \beta)}(x). \quad (2.17)$$

Setting $x = 1$, in (2.17), we obtain

$$\sum_{k=0}^{\infty} b(n, m, k) P_k^{(a, \beta)}(1) h_{a, \beta}(k) = P_n^{(a, \beta)}(1) P_m^{(a, \beta)}(1) \quad (2.18)$$

$$\sum_{k=0}^{\infty} b(n, m, k) \frac{(1+a)k}{k!} h_{a, \beta}(k) = \frac{(1+a)n}{n!} \frac{(1+a)m}{m!}. \quad (2.19)$$

The conjugate function corresponding to (2.17) is

$$b(n^*, m, k) = \int_{-1}^{1} P_m^{(a, \beta)}(-x) P_n^{(a, \beta)}(x) P_k^{(a, \beta)}(x) d\Omega_{\alpha, \beta}(x). \quad (2.20)$$

A result corresponding to (2.18) is

$$b(n, m, k)(x) = P_n^{(a, \beta)}(-x) P_m^{(a, \beta)}(x). \quad (2.21)$$

**Definition 2.2:**

The associated function $f(n, m, \alpha)$ of a function $f(n)$ defined for $n = 0, 1, \ldots$ is given by

$$f(n, m) = \sum_{k=0}^{\infty} f(k) b(n, m, k) h_{a, \beta}(k). \quad (2.22)$$
Definition 2.3:

The conjugate associated function \( f(n^*, m) \) of \( f(n) \) is given by

\[
f(n^*, m) = \sum_{k=0}^{\infty} f(k) b(n^*, m, k) h_{a, \beta}(k).
\] (2.23)

Lemma 2.4:

The associated and conjugate associated functions of \( f(n) \) are given respectively by

\[
f(n, m) = \int \int P^{(a, \beta)}_n(x) P^{(a, \beta)}_m(x) d\Omega_{a, \beta}(x),
\] (2.24)

\[
f(n^*, m) = \int \int P^{(a, \beta)}_n(-x) P^{(a, \beta)}_m(x) d\Omega_{a, \beta}(x).
\] (2.25)

Proof: To prove (2.24) we have

\[
f(n, m) = \sum_{k=0}^{\infty} f(k) b(n, m, k) h_{a, \beta}(k)
\]

\[
= \sum_{k=0}^{\infty} f(k) \int_{-1}^{1} P^{(a, \beta)}_n(x) P^{(a, \beta)}_m(x) P^{(a, \beta)}_k(x) d\Omega_{a, \beta}(x) h_{a, \beta}(k)
\]

\[
= \int_{-1}^{1} P^{(a, \beta)}_n(x) P^{(a, \beta)}_m(x) d\Omega_{a, \beta}(x) \sum_{k=0}^{\infty} f(k) P^{(a, \beta)}_k(x) h_{a, \beta}(k).
\]

In view of (2.6) the proof is complete. The proof of (2.25) is similar to that of (2.24).

Lemma 2.5:

The functions \( f(n, m) \), \( f(n^*, m) \) possess the following properties:

(i) \( f(n, 0) = f(n) \)

(ii) \( f(n, m) = f(m, n) \)

(iii) \( \nabla_n f(n, m) = \nabla_m f(n, m) \)

(iv) \( f(n^*, m) = f(m, n^*) \)

(v) \( \nabla_n f(n^*, m) = - \nabla_m f(n^*, m) \)

(vi) \( [P^{(a, \beta)}_n(\nabla) f](m) = f(n, m) \)

(vii) \( [P^{(a, \beta)}_n(-\nabla) f](m) = f(n^*, m) \).

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The relationship between the Jacobi transform of a function and that of its associated function is given in the following lemma.

**Lemma 2.6:**

Let \( f(x) \) be the Jacobi transform of \( f(n) \). Then the Jacobi transform of \( f(n, m) \), the function associated with \( f(n) \), is

\[
f(., m) \hat{f}(x) = \hat{f}(x) P_{m}^{(a, \beta)}(x). \tag{2.26}
\]

**Proof:** We have

\[
f(., m) \hat{f}(x) = \sum_{n=0}^{\infty} f(n, m) P_{n}^{(a, \beta)}(x) h_{a, \beta}(n)
\]

\[
= \sum_{k=0}^{\infty} f(k) h_{a, \beta}(k) \sum_{n=0}^{\infty} P_{n}^{(a, \beta)}(x) b(n, m, k) h_{a, \beta}(n)
\]

\[
= P_{m}^{(a, \beta)}(x) \sum_{k=0}^{\infty} f(k) P_{k}^{(a, \beta)}(x) h_{a, \beta}(k)
\]

\[
= \hat{f}(x) P_{m}^{(a, \beta)}(x).
\]

Now we shall study functions associated with \( h(n, t) \).

**Lemma 2.7:**

The function associated with \( h(n, t) \) is given by

\[
h(n, m; t) = \int e^{t x} P_{n}^{(a, \beta)}(x) P_{m}^{(a, \beta)}(x) d\Omega_{a, \beta}(x). \tag{2.27}
\]

Evaluating the integral on the right by means of (2, p. 288 (20)) and using the fact

\[
\frac{\Gamma(n - r)}{\Gamma(- r)} = (-1)^{n} \frac{\Gamma(1 + r)}{\Gamma(1 - n + r)}
\]

one can easily show that

\[
h(n, m; t) = (-1)^{n} e^{t} \frac{2^{a+\beta+1}}{m! n!} \Gamma(1 + \beta + n) \Gamma(1 + a + m)
\]

\[
\times \sum_{r=0}^{\infty} \frac{(-2t)^{r}}{\Gamma(1 - n + r) \Gamma(a + \beta + n + 2 + r)}
\]

\[
\times {}_{4}F_{3}\left(\begin{array}{c}
-m, a + \beta + m + 1, 1 + a + r, 1 + r \\
1 + a, a + \beta + n + 2 + r, 1 - n + r
\end{array}; 1 \right) \tag{2.28}
\]
Corollary 2.8:
For $t > 0$,
\[ e^{tx} P_{m}^{(\alpha, \beta)} (x) = \sum_{n=0}^{\infty} h(n, m; t) P_{n}^{(\alpha, \beta)} (x) h_{\alpha, \beta} (m). \]  
\[ (2.29) \]

Lemma 2.9:
The function associated with $h(n^{*}, t)$ is given by
\[ h(n^{*}, m; t) = \sum_{\nu=-1}^{\infty} e^{tx} P_{\nu}^{(\alpha, \beta)} (-x) P_{m}^{(\alpha, \beta)} (x) d\Omega_{\alpha, \beta} (x) \]
\[ (2.30) \]
where
\[ (-2t)^{\nu} \frac{\Gamma(m - r) \Gamma(\beta + m + 1) \Gamma(\beta + n + 1) \Gamma(\alpha + 1 + r)}{m! n! \Gamma(\beta + 1) \Gamma(-r) \Gamma(\alpha + \beta + m + 2 + r)} \]
\[ \times {}_{4}F_{3} \left( -n, \alpha + \beta + n + 1, \alpha + 1 + r, 1 + r \right) \left( \beta + 1, \alpha + \beta + m + 2 + r, 1 - m + r \right). \]  
\[ (2.31) \]
The proof is exactly similar to that of Lemma 2.7.

Corollary 2.10:
For $t > 0$,
\[ e^{tx} P_{n}^{(\alpha, \beta)} (-x) = \sum_{m=0}^{\infty} h(n^{*}, m; t) P_{m}^{(\alpha, \beta)} (x) h_{\alpha, \beta} (m). \]  
\[ (2.32) \]
Since both sides of (2.32) are analytic and since, by the asymptotic estimates of $h(n^{*}, m; t)$ and $P_{m}^{(\alpha, \beta)} (x)$, we can show that the series
\[ \sum_{m=0}^{\infty} h(n^{*}, m, t) P_{m}^{(\alpha, \beta)} (-x) h_{\alpha, \beta} (m) \]
converges absolutely, we have the following additional equality.

Corollary 2.11:
\[ e^{-tx} P_{n}^{(\alpha, \beta)} (x) = \sum_{m=0}^{\infty} h(n^{*}, m; t) P_{m}^{(\alpha, \beta)} (-x) h_{\alpha, \beta} (m). \]  
\[ (2.33) \]

Definition 2.12:
The Poisson Jacobi-transform of a function and defined for $n = 0, 1, \ldots$ is given by
\[ \nu (n, t) = \sum_{m=0}^{\infty} h(n, m; t) \phi (m) h_{\alpha, \beta} (m) \]  
\[ (2.34) \]
whenever the series converges.
Examples of function which are Poisson-Jacobi transform are given in the following table.

1. \[ \frac{v(n, t)}{e^{x^2 P_{n}^{(\alpha, \beta)}(x)}} \]  
   \[ \frac{\phi(m)}{P_{m}^{(\alpha, \beta)}(x)} \]

2. \[ \frac{(1 + \alpha)n}{n!} e^{t} \]  
   \[ \frac{(1 + \alpha)m}{m!} \]

3. \[ \frac{(1 + \alpha)n}{n!} \left[ \frac{n(1 + \alpha + \beta + n) + t}{2(1 + \alpha)} \right] \]  
   \[ \frac{m(1 + \alpha)m(1 + \alpha + \beta + n)}{2m!(1 + \alpha)} \]

3. **Asymptotic Estimates**

In this section, we study the behaviour for large values of the arguments \(m\) and \(n\), of the kernels \(h(\beta, m; t), h(n^*, m; t)\) and related functions.

By Lemma 2.7, we have

\[ h(n, m; t) e^{-t} = (-1)^n \frac{2^{\alpha+\beta+1} \Gamma(1 + \beta + n) \Gamma(1 + \alpha + m)}{n! m! \Gamma(1 + \alpha)} \lim_{N \to \infty} \]

\[ \times \sum_{r=0}^{N} \frac{(-2t)^r \Gamma(1 + \alpha + r) \Gamma(1 - n + r) \Gamma(1 + n + 2 + r)}{\Gamma(1 + \alpha + n + 2 + r) \Gamma(1 + \alpha + \beta + n + 2 + r)} \]

\[ \times \sum_{s=0}^{\infty} \frac{(-m)_s (1 + \alpha + \beta + m)_s (1 + \alpha + r)_s (1 + r)_s}{s! (1 + \alpha)_s (1 + \beta + n + 2 + r)_s (1 - n + r)_s} \]

(3.1)

Using the formulae

\[ \frac{\Gamma(k + a)}{\Gamma(k)} \sim k^a, k \to \infty \]

(3.2)

and

\[ \sqrt{\pi} \Gamma(2z) = 2^{z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) \]

(3.3)

(3.1) can be rewritten as

\[ h(n, m; t) e^{-t} = \frac{\sqrt{\pi} 2^{-\alpha n} \Gamma(1 + \alpha + n)}{m! \Gamma(1 + \alpha)} n^\beta (2t)^n \]

\[ \times \sum_{s=0}^{m} 2^{-s} \frac{(-m)_s (1 + \alpha + \beta + m + 1)_s}{(s!)^2 (1 + \alpha)_s} \]

\[ \times \frac{\Gamma(1 + \alpha + n + s) \Gamma(1 + n + s)}{\Gamma\left(\frac{\alpha + \beta + s}{2} + 1 + n\right) \Gamma[n + \frac{1}{2}(\alpha + \beta + s + 3)] \Gamma(1 + n)} \]

(3.4)
The application of the asymptotic relation

\[
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{a-\beta} [1 + \frac{1}{2}z^{-1}(a - \beta)(a + \beta - 1) + O(z^{-2})]
\]  

leads to the following:

**Theorem 3.1.**

For \( t > 0 \)

\[
h(n, m; t) \sim \frac{\sqrt{\pi 2^{-m-n}}}{m! n!} (1 + \alpha)_m n^{m-1/2} (t^n e^t, \text{ as } n \to \infty)
\]  

**Theorem 3.2:**

For \( t > 0 \),

\[
h(n, m; t) \sim \frac{\sqrt{\pi 2^{-n-m}}}{n! m!} (1 + \alpha)_n m^{n-1/2} (t^m e^t, \text{ as } m \to \infty)
\]  

**Theorem 3.3:**

For \( 0 < t < c \),

\[
h(n, m; t) \sim 0 \left[ \frac{m^{n-1/2}}{m!} \left( \frac{1}{2} c \right)^m \right], \text{ as } m \to \infty
\]  

**Theorem 3.4:**

For the conjugate kernel, we have

\[
h(n^*, m; t) \sim (-1)^n \frac{\sqrt{\pi 2^{-m-n}}}{n! m!} (1 + \beta)_n m^{n-1/2} (t^m e^t, \text{ as } m \to \infty)
\]

and

\[
h(n^*, m; t) \sim (-1)^m 2^{a+\beta+1} \Gamma(\beta + m + 1) e^{-t} n^{-\beta-2} e^{\beta+2m},
\]

as \( n \to \infty \)

An immediate consequence of (3.7) and (3.9) is

\[
\frac{h(n^*, m; t)}{h(n, m; t)} \sim (-1)^n \frac{(1 + \beta)_n}{(1 + \alpha)_n}, \text{ as } m \to \infty
\]
Definition 3.5:

Let us introduce the operators

\[ \psi^+ f(n) = f(n + 1) - f(n) \quad n = 0, 1, \ldots; \]

\[ \psi^- f(n) = f(n) - f(n - 1) \quad n = 0, 1, \ldots; \]

here \( f(-1) \) is to be taken as zero.

Theorem 3.6:

For \( 0 < t < t_0 \),

\[ \frac{\psi^+ (m)}{h(n, m; t)} \frac{h(n, m; t)}{h(n, m; t_0)} = 0 \left[ m^{n-n_0} \left( \frac{t}{t_0} \right)^m \right], \text{ as } m \to \infty, \quad (3.12) \]

Theorem 3.7:

For \( t, t_0 > 0 \),

\[ \frac{\nabla_n h(n, m; t)}{h(n, m; t_0)} = 0 \left[ m \left( \frac{t}{t_0} \right)^m \right], \text{ as } m \to \infty. \quad (3.13) \]

Proof: By definition (2.5) of the operator \( \nabla_n \) and by the asymptotic estimate (3.7), we have

\[ \frac{\nabla_n h(n, m; t)}{h(n, m; t_0)} \sim m \left[ \frac{t}{t_0} \right]^m \]

\[ \left\{ \frac{1}{A_n} \frac{(\alpha + n + 1)}{2(n + 1)} - \frac{B_n}{A_n} + \frac{C_n}{A_n} \frac{2n}{m^{\alpha}(\alpha + n)} \right\} \]

whence (3.13) is immediate.

As a consequence of this theorem, we can state the following result.

Theorem 3.8:

For \( t, t_0 > 0 \) and \( n \) fixed,

\[ \psi^+ (m) \frac{\nabla_n h(n, m; t)}{h(n, m; t_0)} = 0 \left[ m \left( \frac{t}{t_0} \right)^m \right], \text{ as } m \to \infty. \quad (3.14) \]

Theorem 3.9:

For \( t > 0 \),

\[ h(n, n; t) \sim \sqrt{\pi} \Gamma(1 + \alpha) n^{a-3/2} e^{2n} \left( \frac{t}{4n} \right)^n e^t, \text{ as } n \to \infty. \quad (3.15) \]
4. Properties of $h(n, m; t)$ and $h(n^*, m, t)$

In this section, we develop important properties of $h(n, m; t)$ and its conjugate.

In view of corollary (2.8) we have

$$e^{tx} P_n^{(a, b)}(x) = \sum_{m=0}^{\infty} h(n, m; t) P_m^{(a, b)}(x) h_{a, b}(m). \tag{4.1}$$

Multiplying (4.1) by $e^{-t}$, setting $1 - x = z$, expanding $e^{-tz}$ and writing the expansion of the Jacobi Polynomials, we get

$$\frac{(1 + a)_{n}}{n!} \left[ 1 - tz + \frac{t^2z^2}{2} + \ldots \right] \left[ 1 + \frac{-n(1 + a + \beta + n)}{1 + a} \frac{z}{2} + \ldots \right]
\times \frac{n(n - 1)(1 + a + \beta + n)(2 + a + \beta + n)}{2!(1 + a)(2 + a)} \frac{z^2}{4} + \ldots \right]
= e^{-t} \sum_{m=0}^{\infty} h(n, m; t) (1 + a)_{m} \frac{h_{a, b}(m)}{m!} \left[ 1 + \frac{-m(1 + a + \beta + m)}{1 + a} \frac{z}{2} + \ldots \right]
= \frac{m(m - 1)(1 + a + \beta + m)(2 + a + \beta + m)}{2!(1 + a)(2 + a)} \frac{z^2}{4} + \ldots \right](4.2)$$

Equating coefficients of like powers of $z$, we readily establish the following results.

**Theorem 4.1:**

For $n, \ t$ fixed,

$$\sum_{m=0}^{\infty} h(n, m; t) h_{a, b}(m) \frac{(1 + a)_{m} m!}{m!} \frac{(1 + a) n}{n!} e^t, \tag{4.3}$$

$$\sum_{m=0}^{\infty} \frac{m(m - 1)(1 + a + \beta + n)(2 + a + \beta + n)(1 + a)_{m}}{4.2!(1 + a)(2 + a) m!} h(n, m; t) h_{a, b}(m)
= \frac{(1 + a)_{n}}{n!} \left[ \frac{n(1 + a + \beta + n)}{2(1 + a)} - t \right] e^t \tag{4.4}$$

and

$$\sum_{m=0}^{\infty} \frac{m(m - 1)(1 + a + \beta + n)(2 + a + \beta + n)(1 + a)_{m}}{4.2!(1 + a)(2 + a) m!} h(n, m; t) h_{a, b}(m)$$
\[
\frac{(1 + a)n}{n!} e^t \left[ \frac{n(n - 1)(1 + a + \beta + m)(2 + a + \beta + n)}{4.2! (1 + a)(2 + a)} \right] \\
+ \frac{n(1 + a + \beta + n)}{2(1 + a)} t + \frac{t^2}{2}
\]

Equation (4.5)

For the conjugate function, we have corresponding results:

**Theorem 4.2 :**

For \( n, \ t \) fixed,

\[
\sum_{m=0}^{\infty} h(n^*, m; t) \frac{(1 + a)m}{m!} h_{a, \beta}(m) = (-1)^n \frac{(1 + \beta)n}{n!} e^t,
\]

\[
\sum_{m=0}^{\infty} m \frac{(1 + a)m}{m!} \frac{(1 + a + \beta + m)}{2(1 + a)} h(n^*, m; t) h_{a, \beta}(m)
\]

\[
= (-1)^n \frac{(1 + \beta)n}{n!} \left[ \frac{n(n - 1)(1 + a + \beta + n)}{2(1 + \beta)} - t \right] e^t
\]

Equation (4.6)

and

\[
\sum_{m=0}^{\infty} \frac{(1 + a)m}{m!} \frac{m(m + 1)(1 + a + \beta + m)(2 + a + \beta + m)}{4.2!(1 + a)(2 + a)}
\]

\[
h(n^*, m; t) h_{a, \beta}(m)
\]

\[
= (-1)^n \frac{(1 + \beta)n}{n!} e^t \left[ \frac{n(n - 1)(1 + a + \beta + n)}{4.2!(1 + \beta)(2 + \beta)} \right] \\
+ \frac{n(1 + a + \beta + n)t}{2(1 + \beta)} + \frac{t^2}{2}
\]

Equation (4.7)

These results can be easily deduced from those given in the preceding theorem on using the relation

\[
P_n^{(a, \beta)}(-x) = (-1)^n P_n^{(a, \beta)}(x),
\]

Equation (4.9)

From the orthogonality relation (2.2) and the integral representation (2.25) the following result can be readily established by using Lebesgue's dominating convergence theorem.

**Theorem 4.3 :**

For \( n, \ m \) fixed

\[
\lim_{t \to 0^+} h(n, m; t) = \frac{\delta(n, m)}{h_{a, \beta}(n)}
\]

Equation (4.10)

It can be generalized in the following form:
Corollary 4.4:

For \( t > 0, n \) fixed,

\[
\lim_{t \to 0^+} \sum_{m=k_1}^{k_2} h(n, m; t) = \frac{1}{h_{a,\beta}(n)}, \quad 0 \leq k_1 < n < k_2 \leq \infty \tag{4.11}
\]

\[
= 0, \quad 0 \leq k_1 \leq k_2 < n \leq \infty \tag{4.12}
\]

\[
= 0, \quad 0 \leq n < k_1 \leq k_2 \leq \infty. \tag{4.13}
\]

It is easy to show that

\[
\frac{\partial h(n, m; t)}{\partial t} = \nabla_n h(n, m; t) \tag{4.14}
\]

and

\[
\frac{\partial h(n^*, m; t)}{\partial t} = \nabla_n h(n^*, m; t). \tag{4.15}
\]

Hence we have the following result.

Theorem 4.5:

The kernels \( h(n, m; t) \) and \( h(n^*, m; t) \) are solutions of the Jacobi difference heat equation

\[
\nabla_n v(n, t) = \frac{\partial v(n, t)}{\partial t}. \tag{4.16}
\]

The function \( h(n, m; t) \) satisfies the following important property.

Theorem 4.6:

For \( n, m \) fixed,

\[
\sum_{k=0}^{\infty} h(k, m; t_1) h(n, k; t_2) h_{a,\beta}(k) = h(n, m; t_1 + t_2). \tag{4.17}
\]

Proof: We have

\[
\sum_{k=0}^{\infty} h(k, m; t_1) h(n, k; t_2) h_{a,\beta}(k) = \frac{1}{h_{a,\beta}(n)} \int_{-1}^{1} e^{t_1 x} P_{k}(a, \beta)(x) P_{n}(a, \beta)(x) d\Omega_{a,\beta}(x)
\]

\[
= \frac{1}{h_{a,\beta}(n)} \int_{-1}^{1} e^{t_1 x} P_{n}(a, \beta)(x) \left[ \sum_{k=0}^{\infty} h(n, k; t_2) P_{k}(a, \beta)(x) h_{a,\beta}(k) \right] d\Omega_{a,\beta}(x),
\]
where term by term integration can be easily justified. Now by (2.29) it follows that

\[
\sum_{k=0}^{\infty} h(n, k; t_1) h(n, k; t_2) h_{\alpha\beta}(k)
= \int_{-1}^{1} e^{(t_1 + t_2)x} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) d\Omega_{\alpha\beta}(x)
\]

and the theorem is proved.

A companion result which may be established similarly is the following:

**Theorem 4.7:**

For \( n, m \) fixed,

\[
\sum_{k=0}^{\infty} h(n^*, k; t_1) h(k^*, m; t_2) h_{\alpha\beta}(k) = h(n, m; t_2 - t_1).
\]

**Proof:** We have

\[
\sum_{k=0}^{\infty} h(n^*, k; t_1) h(k^*, m; t_2) h_{\alpha\beta}(k)
= \sum_{k=0}^{\infty} h(n^*, k; t_1) h_{\alpha\beta}(k) \int_{-1}^{1} e^{t_2x} P_k^{(\alpha, \beta)}(-x) P_m^{(\alpha, \beta)}(x) d\Omega_{\alpha\beta}(x)
= \int_{-1}^{1} e^{t_2x} P_m^{(\alpha, \beta)}(x) \left[ \sum_{k=0}^{\infty} P_k^{(\alpha, \beta)}(-x) h(n^*, k; t_1) \right] d\Omega_{\alpha\beta}(x).
\]

Now by (2.33), it follows that

\[
\sum_{k=0}^{\infty} h(n^*, k; t_1) h(k^*, m; t_2) h_{\alpha\beta}(k)
= \int_{-1}^{1} e^{(t_1 - t_2)x} P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) d\Omega_{\alpha\beta}(x)
\]

and the theorem is proved.

5. **Convergence**

In this Section, we determine the convergence conditions for the Poisson-Jacobi transform. Further, we show that the convergence of the transform implies that of the corresponding conjugate transform.
**Poisson–Jacobi transforms**

**Theorem 5.1:**

For $m, m_0$ fixed, $m < m_0$ and $0 < t < t_0$

$$\lim_{n \to \infty} \frac{h(n, m; t)}{h(n_0, m; t_0)} = 0.$$  \hspace{1cm} (5.1)

The proof is immediate in view of theorem 3.1.

**Definition 5.2:**

Let us define

$$s(m) = \frac{h(n, m; t)}{h(n_0, m; t_0)}.$$  \hspace{1cm} (5.2)

Then as $m \to \infty$,

$$s(m) = 2^{n-n_0} \frac{n! \Gamma(1 + \alpha + \eta)}{n_0! \Gamma(1 + \alpha + n_0)} m^{n-n_0} \left( \frac{t}{t_0} \right)^m e^{t-t_0} = 0, \text{ if } n < n_0. \hspace{1cm} (5.3)$$

**Theorem 5.3:**

Let $\phi(m)$ be a real valued function defined for $m = 0, 1, \ldots$ If the Poisson–Jacobi transform

$$v(n, t) = \sum_{m=0}^{\infty} h(n, m; t) \phi(m) h_{\alpha, \beta}(m)$$  \hspace{1cm} (5.4)

converges conditionally for $(n_0, t_0), 0 \leq n_0 < \infty, 0 < t_0 < 1$, then it converges for all $n = 0, 1, \ldots$ and $0 < t \leq t_0 < 1$.

**Proof:** We have

$$v(n, t) = \sum_{n=0}^{\infty} s(m) h(n_0, m, t_0) \phi(m) h_{\alpha, \beta}(m)$$

where $s(m)$ is defined by (5.2). Since $\sum_{n=0}^{\infty} h(n_0, m, t_0) \phi(m) h_{\alpha, \beta}(m)$ converges by hypothesis, whereas $s(m) > 0$, and since $s(m)$ for large $m$, the conclusion holds by an elementary theorem based on partial summation. 

See (8, p. 316).

For the conjugate Poisson–Jacobi transform, we have the following result:

**Theorem 5.4:**

If

$$\psi(n^*, t) = \sum_{m=0}^{n^*} h(n^*, m; t) \phi(m) h_{\alpha, \beta}(m)$$  \hspace{1cm} (5.5)
converges absolutely for \( t = t_0, \ 0 < t_0 < 1 \), then it converges absolutely for all \( t, \ 0 < t \leq t_0 \leq 1 \).

**Proof:** We have

\[
I = \sum_{n=0}^{\infty} |h(n^*, m; t)| |\phi(m)| h_{a, \beta}(m)
\]

\[
= \sum_{m=0}^{\infty} \left| \frac{h(n^*, m; t)}{h(n^*, m; t_0)} \right| |h(n^*, m; t_0)| |\phi(m)| h_{a, \beta}(m).
\]

By (3.9) \( h(n^*, m; t)/h(n^*, m; t_0) \) is a bounded function of \( m \), and hence the result.

**Corollary 5.5:**

If

\[
v(n, t) = \sum_{n=0}^{\infty} h(n, m; t) \phi(m) h_{a, \beta}(m)
\]

converges absolutely for \( t = t_0, \ 0 \leq t_0 \leq 1 \), then so does

\[
v(n^*, t) = \sum_{m=0}^{\infty} h(n^*, m; t) \phi(m) h_{a, \beta}(m)
\]

for \( 0 < t \leq t_0 \leq 1 \).

**Proof:** We have

\[
I = \sum_{m=0}^{\infty} |h(n^*, m; t_0)| |\phi(m)| h_{a, \beta}(m)
\]

\[
= \sum_{m=0}^{\infty} \left| \frac{h(n^*, m; t_0)}{h(n, m; t_0)} \right| |h(n, m; t_0)| |\phi(m)| h_{a, \beta}(m).
\]

Using (3.11), we have

\[
I \leq k \sum_{m=0}^{\infty} |h(n, m; t_0)| \phi(m)| h_{a, \beta}(m).
\]
Since the series converges by hypothesis, we have the absolute convergence of
\[ \sum_{n=0}^{\infty} h(n^*, m ; t_0) \phi(m) h_{\alpha, \beta}(m) \]
and the conclusion of the corollary follows from the preceding theorem.

6. Inversion

Now, we establish an inversion theorem for the Poisson-Jacobi transform. We shall also obtain a conjugate inversion formula.

Theorem 6.1:

Let \( \phi(n) \) be a function defined for \( n = 0, 1, 2, \ldots \) and let
\[ Z h(n, m; t) \phi(m) h_{\alpha, \beta}(m) \] (6.1)
converge for \( (n_0, t_0), 0 \leq n_0 < \infty, 0 \leq t_0 \leq 1 \). Then
\[ \lim_{t \to 0^+} \sum_{m=0}^{\infty} h(n, m; t) \phi(m) h_{\alpha, \beta}(m) = \phi(n). \] (6.2)

Proof: Let
\[ q(k) = \sum_{m=k}^{\infty} h(n_0, m; t_0) \phi(m) h_{\alpha, \beta}(m) \] (6.3)
and for \( 0 < t < t_0 \leq 1 \), let
\[ s(k) = \frac{h(n, k; t)}{h(n_0, k; t_0)}. \] (6.4)

By the convergence of the series (6.1), we note that, given \( \epsilon > 0 \), there exists a positive number \( N_1 \), such that
\[ |q(k)| < \epsilon, \quad k \geq N_1. \] (6.5)

Further, there exists a positive number \( N_2 \) such that, for \( t \) sufficiently small,
\[ s(k) \downarrow, \quad K \geq N_2. \] (6.6)

Let
\[ N = \text{Max} \{N_1, N_2, n + 1\}. \]
Now, we have

\[ I = \sum_{m=0}^{\infty} h(n, m; t) \phi(m) h_a, \beta(m) \]

\[ = \sum_{m=0}^{N-1} h(n, m; t) \phi(m) h_a, \beta(m) + \sum_{m=N}^{\infty} h(n, m; t) \phi(m) h_a, \beta(m) \quad (6.7) \]

\[ = I_0 + I_2. \quad (6.8) \]

It is clear by (4.10) that

\[ \lim_{t \to 0} I_1 = \phi(n). \]

Further, we have

\[ I_2 = \sum_{m=N}^{\infty} h(n, m; t) \phi(m) h_a, \beta(m) \]

\[ = \sum_{m=N}^{\infty} s(m) h(n_0, m; t_0) \phi(m) h_a, \beta(m) \]

\[ = \sum_{m=N}^{\infty} s(m) [q(m) - q(m + 1)] \quad (6.9) \]

\[ = q(N) s(N) + \sum_{m=N+1}^{\infty} q(m) [s(m) - s(m - 1)]. \]

Hence

\[ |I_2| \leq |q(N)| \cdot |s(N)| + \sum_{m=N+1}^{\infty} |q(m)| \cdot |s(m) - s(m - 1)| \]

\[ \leq \epsilon \cdot |s(N)| + \epsilon \cdot \sum_{m=N+1}^{\infty} |s(m) - s(m - 1)| \]

\[ \leq 2 \epsilon \cdot |s(N)|, \quad (6.10) \]

and therefore

\[ \lim_{t \to 0^+} |I_2| \leq 0, \quad (6.11) \]

so that the proof is complete.
The conjugate inversion formula is as follows:

**Theorem 6.2:**

Let

\[ v(n, t) = \sum_{m=0}^{\infty} h(n, m; t) \phi(m) h_{\alpha,\beta}(m) \]  \hspace{1cm} (6.12)

converge absolutely for \((n_0, t_0), 0 < n_0 < \infty, 0 < t_0 \leq 1.\) Then

\[ \phi(n) = \sum_{m=0}^{\infty} h(n^*, m; t) v(m^*, t) h_{\alpha,\beta}(m) \]  \hspace{1cm} (6.13)

for \(0 < t \leq t_0 \leq 1.\)

**Proof:** By corollary (5.5) we have

\[ v(n^*, t) = \sum_{m=0}^{\infty} h(n^*, m; t) \phi(m) h_{\alpha,\beta}(m) \]

for \(0 < t \leq t_0 \leq 1,\) with the series converging absolutely. Hence for \(0 < t < t' \leq t_0,\) it follows that

\[ \sum_{m=0}^{\infty} h(n^*, m; t) v(m^*; t') h_{\alpha,\beta}(m) \]

\[ = \sum_{m=0}^{\infty} h(n^*, m; t) h_{\alpha,\beta}(m) \sum_{k=0}^{\infty} h(m^*, k; t') \phi(k) h_{\alpha,\beta}(k) \]

\[ = \sum_{k=0}^{\infty} \phi(k) h_{\alpha,\beta}(k) \sum_{m=0}^{\infty} h(n^*, m; t) h(m^*, k; t') h_{\alpha,\beta}(m) \]

\[ = \sum_{k=0}^{\infty} h(n, k; t' - t) \phi(k) h_{\alpha,\beta}(k) \]

by appealing theorem 4.7.

Also by preceding theorem, we have

\[ \phi(n) = \lim_{t \to t'} \sum_{m=0}^{\infty} h(n^*, m; t) v(m^*, t') h_{\alpha,\beta}(m) \]

The series on the right is a continuous function of \(t\) for \(0 < t \leq t',\) and so

\[ \phi(n) = \sum_{m=0}^{\infty} h(n^*, m; t') v(m^*, t') h_{\alpha,\beta}(m) \]

\(0 < t' \leq t_0,\) and the theorem is established.
REFERENCES


