On the line of striction and the parameter of distribution of
the generators of a scroll

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1. The object of the present paper is to examine the existence of
and determine, when possible, scrolls "with a given line of striction and
parameter of distribution ($\beta$) of the generators. A differential equation of
the second order is obtained whose coefficients involve both dependent and
independent variables and which does not appear to be solvable in the
general case. Some particular cases are considered and some surfaces with
interesting properties are obtained. The notations used are those in

2. Let $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be the orthogonal trihedral associated with any point
on the line of striction, $\mathbf{d}$ the unit generator there, $\nu$ the angle which $\mathbf{d}$ makes
with $\mathbf{t}$. Then because $\mathbf{d}$ is perpendicular to both $\mathbf{d}$ and $\mathbf{t}$, we have

$$\mathbf{d}' = (\mathbf{d} \times \mathbf{t})/\beta$$

$$= a (\mathbf{n} \cos \nu + \mathbf{b} \sin \nu), \quad \text{where} \quad a = |\mathbf{d}'|.$$ 

Taking

$$\mathbf{d} = \mathbf{t} \cos \nu - \sin \nu (\mathbf{n} \sin \nu - \mathbf{b} \cos \nu),$$

we have

$$\mathbf{d}' = \mathbf{t} \sin \nu \left( k \sin \nu - \frac{d\nu}{ds} \right)$$

$$+ \mathbf{n} \left\{ k \cos \nu - \cos \nu \sin \nu \frac{d\nu}{ds} - \left( \tau + \frac{d\omega}{ds} \right) \sin \nu \cos \omega \right\}$$

$$+ \mathbf{b} \left\{ \cos \nu \cos \omega \frac{d\omega}{ds} - \sin \nu \sin \left( \tau + \frac{d\omega}{ds} \right) \right\}$$

$$\therefore \frac{d\nu}{ds} = k \sin \nu$$

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and then
\[ d' = \{k \cos v \cos^2 w - \left(\tau + \frac{dw}{ds}\right) \sin v \cos w\} \bar{n} \]
\[ + \{\cos v \cos w \sin w - \left(\tau + \frac{dw}{ds}\right) \sin v \cos w\} \bar{b} \]
\[ = \{k \cos v \cos w - \left(\tau + \frac{dw}{ds}\right) \sin w\} \{\bar{n} \cos w + \bar{b} \sin w\} \]
\[ \therefore a = k \cos v \cos w - \left(\tau + \frac{dw}{ds}\right) \sin v \]

or
\[ \frac{1}{\beta} + \tau + \frac{dw}{ds} = k \cot v \cos w \therefore \beta = \sin v/a \]

or
\[ \cot v = b \sec w + \rho \sec w \frac{dw}{ds}, \text{ where } b = \rho \left(\frac{1}{\beta} + \tau\right). \]

Hence we have the two equations for finding \( v \) and \( w \)
\[ \frac{dv}{ds} = k \sin \omega \]  \hspace{1cm} (1)

and
\[ \cot v = b \sec w + \rho \sec w \frac{dw}{ds} \]  \hspace{1cm} (2)

3. Differentiating (2) and eliminating \( v \) we have
\[ \sec^2 w \frac{d^2 w}{ds^2} + 2 \sec^2 w \tan w \left(\frac{dw}{ds}\right)^2 + (3b/\rho \tan w + \rho'/\rho) \]
\[ \times \sec^2 w \frac{dw}{ds} + (b'/\rho + b^2/\rho^2 \tan w + \sin w \cos w/\rho^2) \]
\[ \times \sec^2 w = 0. \]  \hspace{1cm} (3)

Putting
\[ \tan w = z, \]

we have the second order differential equation
\[ \frac{d^2 z}{ds^2} + (3b/\rho z + \rho'/\rho) \frac{dz}{ds} + (b'/\rho + b^2z/\rho^2) (1 + z^2) + z/\rho^2 = 0 \]  \hspace{1cm} (4)
whose solution will contain two arbitrary constants. Hence we have the

corem:—

*There exists a doubly infinite family of scrolls having a given line of
striction and parameter of distribution.*

4. The equation (4) is however not solvable on the general case and
we shall examine some particular cases.

Taking

\[ b = \rho \left( \frac{1}{\beta + \tau} \right) = c, \]

a constant, we have from (2)

\[ \cot v \cos w - c = \rho \frac{dw}{ds} \]

which with the help of (1) can be written as

\[ (\cot v \cos w - c) \frac{dv}{ds} = \sin w \frac{dw}{ds}, \]

or

\[ \frac{dy}{ds} + y \cot v \frac{dv}{ds} = c \frac{dv}{ds}, \]

where

\[ y = \cos w \]

Multiplying by \( \sin v \) and integrating

\[ y \sin v = p - c \cos v, \] where \( p \) = an arbitrary constant

or

\[ \sin v \cos w + c \cos v = p. \]

This serves as an intermediate integral and gives

\[ \cos w = p \cosec v - c \cot v. \]

Hence from (1) we have

\[ \rho \sin v \frac{dv}{ds} = \sqrt{1 - p^2 + 2pc \cos v + (c^2 - 1) \cos^2 v}. \]

Putting \( \cos v = x \), we have

\[ \int \frac{-dx}{\sqrt{1 - p^2 + 2pcx + (c^2 - 1)x^2}} = \int \frac{ds}{\rho} = d\psi \] (6)
This gives \( v \); \( w \) can then be determined from (5) and the doubly infinite scrolls obtained.

If \( c = 0 \), the equation (6) gives

\[
\cos v = \sqrt{1 - p^2} \cos (\psi + q),
\]

where \( q \) is an arbitrary constant. We can then get \( w \) from

\[
\cos w = p / \sin v.
\]

5. When only the line of striction is given and not also \( \beta \), we have only one relation, (1), between the quantities \( v \) and \( w \) and we can make the ray satisfy another condition. If for example the rays are to intersect a given curve \( \Gamma \) whose coordination are functions of a parameter \( T \), \( \vec{R} = \vec{R}(T) \), we have, in order that the ray may meet \( \Gamma \)

\[
\vec{R} = \vec{r} + u \vec{d}.
\]

This gives, on elimination of \( u \), two equations in \( T, s, v \) and \( w \). Eliminating \( T \) between these equations we have a relation of the form

\[
f(s, v, w) = 0
\]

We can then determine \( v \) and \( w \) as functions of \( s \) from equations (1) and (7) which will involve one arbitrary constant. The parameter of distribution will be such as to satisfy the equation (2). Hence we have the result:

There exists a singly infinite family of scrolls having a given line of striction so that the generators meet another given curve.

6. When the given line of striction is a straight line, which has no definite principal normal or binormal, we have because \( k = 0 \), \( v = a \) constant. (a) which varies from member to member. Also because the line of striction can be regarded as an asymptotic line on the scroll, + , the normal to the surface its binormal and therefore \( w = \pi/2 \). The parameter of distribution instead of being pre-assigned, must be such that \( 1/\beta = -\tau = -\sqrt{-K} \), where \( K \) is the Gaussian curvature. To fix the generator we require a second angle.

Taking the given line as the \( Z \)-axis, the equation of the ray through \( (0, 0, z) \) making an angle \( a \) with it can be taken as

\[
\frac{X}{\cos \theta \sin a} = \frac{Y}{\sin \theta \sin a} = \frac{Z - z}{\cos a} = u.
\]
If the line is to satisfy another condition, e.g., meeting a given curve or touching a given surface, the parameters \((z, \theta)\) are connected by a relation so that \(z\) can be regarded as a function of \(\theta\). Taking \((u, \theta)\) to be the curvilinear coordinates of any point on the surface, we have, on calculating the fundamental magnitudes

\[-K = z'^2/(u^2 + z'^2)\]

\[\therefore \tau = \sqrt{-K} = z'/(u^2 + z'^2)\]

As on the \(z\)-axis \(\tau = 1/z'\), the parameter of distribution can be taken to be \(-z'\).

7. Taking the given curve to be a straight line at right angles to the \(z\)-axis given by

\[Z = 0, \quad lX + n \tan \alpha = 0, \tag{9}\]

the condition that the line (8) meets (9) is given by

\[Z \pm n/l \cos \theta. \tag{10}\]

Eliminating \(\theta\) and \(Z\) we have the equation of the scroll

\[X^2 Z^2 l^2 \tan^2 \alpha = (X^2 + Y^2) (lX + n \tan \alpha)^2 \tag{11}\]

For a given value of \((\theta)\) there is only one value of \(Z\) (\(|Z| > |n/l|\)) but for a given value of \(Z\), there are, in general, two values of \((\theta)\).

Hence through any point on the straight line of striction there pass two generators of the scroll.

Again any plane through the given line (9)

\[lX + n \tan \alpha = \lambda Z \tan \alpha \tag{12}\]

meets the surface where

\[Z^2 = 0 \quad \text{or} \quad l^2 X^2 = (X^2 + Y^2)\]

which represents two planes through the \(Z\)-axis. These intersect the plane (12) in two straight lines through the point \((0, 0n/\lambda)\).

Hence the given line is a double line on the scroll. Any plane through the \(Z\)-axis meets the scroll in two straight line through the point where the planes meet (9).
8. We next consider the case when rays from points on the straight line of striction (the Z-axis) meets a conic C whose plane meets the line at O. If \( m, M \) (possibly \( \infty \)) are the minimum and maximum radii vectors from O to C, the points on the Z-axis from which rays can be drawn making an angle \( \alpha \) with it, to meet C, must be situated between two points whose distances from O depend on \( m \) and \( M \). Corresponding to any point \( p \) on C, there is only one point \( p \) on the Z-axis but corresponding to any point \( p \) on the Z-axis there are four points \( p_1, p_2, p_3, p_4 \) (real or imaginary) on C, these being the intersection of C with the right circular cone having \( P \) as vertex, Z-axis as the axis and semivertical angle \( \alpha \). The Z-axis is thus a singular line. Any plane through the Z-axis meets the scroll in two parallel lines, making an angle \( \alpha \) with the Z-axis, meeting C where the plane meets it. The surface generated is of the 8th degree.