On the coefficients of a bounded univalent analytic function

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ABSTRACT

Let \( F_M \) denote the class of univalent analytic functions \( f \) in \( |z| < 1 \) with the expansion \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) and \( |f(z)| < M \) in \( |z| < 1 \). In this note I derive a rough bound for all \( n \)-th coefficients and a more accurate bound for all the third coefficients of functions \( f \) belonging to \( F_M \).

Let \( F \) denote the class of all the analytic functions \( f \) which are univalent in \( |z| < 1 \) with \( f(0) = 0 \) and \( f'(0) = 1 \). That is \( f \) has a Taylor expansion

\[
f(z) = z + a_2z^2 + a_3z^3 + \cdots
\]

in the unit disc. The famous Bieberbach's conjecture\(^1\) states that for any \( f \in F \),

\[
|a_n| \leq n
\]

for all the integers \( n \). So far, the conjecture has been verified for \( n \leq 6 \) (see Pederson\(^2\)). Recently, Fitzgerald\(^3\) showed that for any \( f \in F \),

\[
|a_n| < \frac{\sqrt{7}}{\sqrt{6}} n
\]

for all the integers \( n \geq 2 \). It is, then, very natural for one to expect that, to each \( n \), there is a more stringent bound for the \( n \)-th coefficients of all the functions of a subclass of \( F \) which is bounded uniformly in the unit disc. The purpose of this note is to present such a study.

For the convenience of the latter discussion, I shall use \( F_M \) to denote the class of all the functions \( f \in F \) with \( |f(z)| \leq M \) in the disc.
**THEOREM 1.** Let \( f(z) = z + a_1 z^2 + \cdots + a_n z^n + \cdots \in F_M \), then

\[
|a_n| \begin{cases} 
\leq n - \frac{1}{nM^n} & \text{if } 2 \leq n \leq 6, \\
< \sqrt{7} n/\sqrt{6} - \frac{1}{nM^n} & \text{if } n > 6.
\end{cases} (4)
\]

**Proof.** Let

\[
f_\lambda(z) = f(z) + \frac{e^{i\lambda} f(z)^n}{nM^n}, \lambda \text{ real.} (5)
\]

Then, it is easy to verify that \( f_\lambda \) is analytic and univalent in \( |z| < 1 \) with \( f_\lambda(0) = 0 \) and \( f_\lambda'(0) = 1 \). That is, \( f_\lambda \in F \) and has the power series expansion

\[
f_\lambda(z) = z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + \left( a_n + \frac{e^{i\lambda}}{nM^n} \right) z^n + \text{higher powers of z.} (6)
\]

It therefore, follows that

\[
|a_n + \frac{e^{i\lambda}}{nM^n}| \begin{cases} 
\leq n & \text{if } 2 \leq n \leq 6, \\
< \sqrt{7} n/\sqrt{6} & \text{if } n > 6.
\end{cases} (7)
\]

Since \( \lambda \) can be any real number, it follows from the above that

\[
|a_n| + \left| \frac{1}{nM^n} \right| \begin{cases} 
\leq n & \text{if } 2 \leq n \leq 6, \\
< \sqrt{7} n/\sqrt{6} & \text{if } n > 6.
\end{cases}
\]

This also proves the assertion.

By considering a different auxiliary function and combining the result thus obtained with (4), one can derive a better bound for \( a_3 \) as follows:

**THEOREM 2.** If \( f(z) = z + a_1 z + a_2 z^2 + \cdots \in F_M \), then

\[
|a_3| \begin{cases} 
\leq \text{Min} \left\{ 3 - \frac{1}{3M^3}, 3 - \left( \frac{2}{M^2} - \frac{1}{M} \right) \right\} & \text{if } |a_2| \geq \frac{1}{2M}, \\
\leq \text{Min} \left\{ 3 - \frac{1}{3M^3}, 3 - \left( \frac{1}{M^2} - \frac{2}{M} \right) \right\} & \text{if } |a_2| < \frac{1}{2M}.
\end{cases} (8)
\]

**Remark.** A sharp bound for \( a_3 \) should be \( 2 - \frac{2}{M} \) which is due to
Proof of Theorem 2. Let

\[ f_\beta (z) = \frac{M f(z)}{M - f(z) e^{i\beta}}, \beta \text{ real.} \]  \hfill (9)

Then it is easy to check that \( f_\beta \in F \) and

\[
\begin{align*}
\left( f_\beta (z) \right) & = f(z) \left( 1 - \frac{e^{i\beta}}{M} f(z) \right)^{-1} \\
& = f(z) \left( 1 + \frac{e^{i\beta}}{M} f(z) + \frac{e^{2i\beta}}{M^2} f^2(z) + \cdots \right) \\
& = f(z) + \frac{e^{i\beta}}{M} f^2(z) + \frac{e^{2i\beta}}{M^2} f^3(z) + \cdots \\
& = z + \left( a_3 + \frac{e^{i\beta}}{M} \right) z^2 + \left( a_3 + \frac{2a_2 e^{i\beta}}{M} + \frac{e^{2i\beta}}{M^2} \right) z^3 + \\
& \quad \text{higher powers of } z.
\end{align*}
\]

Using the result (2) for \( n = 3 \), we have

\[
| a_3 + \frac{2a_2 e^{i\beta}}{M} + \frac{e^{2i\beta}}{M^2} | \leq 3. \tag{10}
\]

Assertion (8) follows from this by a proper choice of \( \beta \).

Corollary. If \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in F_M \) with both \( a_2, a_3 \) being real and nonnegative or \( a_2 \geq 0, a_2 \leq 0 \), then

\[
| a_3 | \leq 3 - \frac{2 | a_2 |}{M} - \frac{1}{M^3}. \tag{11}
\]

In the proof of Theorem 2 if one follows the argument used by Nehari (p. 224, ex. 4) by considering the auxiliary function

\[
f_\lambda (z) = \frac{f(z)}{(1 + e^{i\lambda} f(z) / M)^2}
\]

\( \lambda \) real instead of (9), then one can derive an improved upper bound for \( a_3 \) as follows.

\[
| a_3 | \begin{cases} \leq \text{Min} \left\{ 3 - \frac{1}{3M^3}, 3 - \left( \frac{4 | a_2 |}{M} - \frac{3}{M^2} \right) \right\} & \text{if } | a_2 | \geq \frac{3}{4M}, \\
\leq \text{Min} \left\{ 3 - \frac{1}{3M^3}, 3 - \left( \frac{3}{M^2} - \frac{4 | a_2 |}{M} \right) \right\} & \text{if } | a_2 | < \frac{3}{4M}. \end{cases} \tag{12}
\]
It is known \(^6\) that
\[
|a_2^2 - a_3^2| \leq 1. \tag{13}
\]
From this and the estimates (12) and (4) one can easily derive the following more accurate bound for \(a_3\).

**Theorem 3.** Let \(f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in F_M\), then

\[
|a_3| \leq \min \left\{ 3 - \frac{1}{3M^2}, |a_3^*| \right\}
\]
where

\[
\begin{align*}
|a_3^*| &= 3 + \frac{3}{M^2} - \frac{4 |a_2|}{M} \quad \text{if} \quad |a_2| > \frac{3}{4M}; M \geq \sqrt{2} , \\
|a_3| &= 1 + |a_2|^2 \quad \text{if} \quad \frac{3}{4M} \leq |a_2| \leq \frac{2}{3M} + \frac{\sqrt{2} - 7M^2}{M} ; M \leq \sqrt{\frac{121}{89}} , \\
|a_3^*| &= 3 + \frac{4 |a_2|}{M} - \frac{3}{M^2} \quad \text{if} \quad \frac{2}{3M} < |a_2| < \frac{3}{4M} , \\
|a_3| &= 3 + \frac{2 |a_2|}{M} \quad \text{if} \quad \frac{1}{2M} < |a_2| < \frac{2}{3M} , \\
|a_3^*| &= 3 + \frac{4 |a_2|}{M} - \frac{3}{M^2} \quad \text{if} \quad |a_2| \leq \frac{1}{2M} .
\end{align*}
\]

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**References**


