ON ALMOST CONTACT METRIC HYPERSURFACE OF A KAHLER MANIFOLD

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ABSTRACT

We have studied the normality of an almost contact metric hypersurface of a Kahler manifold. Quasi-umbilical and umbilical properties have also been studied.

1. INTRODUCTION

An $n$-dimensional real differentiable manifold $M_n$ of differentiability class $C^\infty$ is said to be an almost contact manifold with the structure $- (F, T, A)$ if

$\tilde{X} + X = A(X)T; \quad \tilde{X} \overset{\text{def}}{=} F(X)$

(1.1)

where $F$ is a linear, vector valued function, $A$ is a 1-form and $T$ is a $C^\infty$ vector field in $M_n$.

From (1.1), we have

(a) $\text{rank}(F) = n - 1$, (b) $A(T) = 1$, (c) $T = 0$, (d) $n$ is odd say $2m + 1$, (e) $A(\tilde{X}) = 0$, for arbitrary vector field $X$.

(1.2)

An almost contact manifold $M_n$ is said to be an almost contact metric manifold if a Riemannian metric $g$ satisfies

$g(\tilde{X}, \tilde{Y}) = g(X, Y) - A(X)A(Y)$

(1.3)

for arbitrary vector field $X, Y$ in $M_n$. The structure $- (F, T, A, g)$ is called an almost contact metric structure. If $N$ be the Nijenhuis tensor of $F$:

$N(X, Y) = [\tilde{X}, \tilde{Y}] + [X, Y] - [\tilde{X}, Y] - [X, \tilde{Y}]$

(1.4)
An almost contact manifold is said to be normal if
\[ N(X, Y) + dA(X, Y) = 0. \] (1.5)

Let \( K \) be the curvature tensor in \( M_n \). The manifold \( M_n \) is said to be of constant sectional curvature if
\[ \mathcal{K}(X, Y, Z, U) = k \{ g(X, U)g(Y, Z) - g(Y, U)g(X, Z) \}, \] (1.6)
where \( \mathcal{K} \) is the Riemannian curvature tensor of type \((0, 4)\) in \( M_n \).

### 2. Almost Contact Metric Hypersurface

Let us assume that \( M_{n+1} \) be a Kahler manifold with the structure \((J, G)\).

Then
\[ J(J(\tilde{X})) = -\tilde{X} \] (2.1)
\[ G(J\tilde{X}, J\tilde{Y}) = G(\tilde{X}, \tilde{Y}) \] (2.2)
\[ (E\tilde{X}J)(\tilde{Y}) = 0 \] (2.3)

for arbitrary vector fields \( \tilde{X}, \tilde{Y} \) in \( M_{n+1} \), where \( E \) is a Riemannian connexion in \( M_{n+1} \). Let \( M_n \) be a hypersurface of \( M_{n+1} \) with \( B \) as its Jacobian map.

Then
\[ G(BX, BY) = g(X, Y) \] (2.4)
where \( g \) is the induced metric of \( M_n \). If \( D \) is the induced Riemannian connexion on \( M_n \), we have
\[ E_{\tilde{X}}BY = BD\tilde{X}Y + h(X, Y)M \] (2.5)
\[ E_{\tilde{X}}M = -BHX, \]
where \( h \) is second fundamental form, \( M \) is a unit normal vector to \( M_n \) and \( H \) is the third fundamental tensor:
\[ g(HX, Y) = h(X, Y). \]

The Gauss and Codazzi equations are given by
\[ \mathcal{K}(BX, BY, BZ, BU) \]
\[ = \mathcal{K}(X, Y, Z, U) - h(X, U)h(Y, Z) + h(X, Z)h(Y, U) \] (2.7)
\[ \mathcal{R}(BX, BY, BZ, M) = (D_xh)(Y, Z) - (D_yh)(X, Z) \] (2.8)
where \( R \) is the Riemannian curvature tensor of type \((0, 4)\) with respect to the connexion \( E \) in \( M_{n+1} \).

If there exist two functions \( \alpha \) and \( \beta \) and a 1-form \( u \) on a hypersurface \( M_n \) such that
\[
h(X, Y) = \alpha g(X, Y) + \beta u(X) u(Y),
\]
\( M_n \) is said to be quasi-umbilic in the normal direction \( M \). In particular, if \( \alpha \) vanishes identically \( M_n \) is said to be cylindrical and if \( \beta \) vanishes identically \( M_n \) is said to be umbilical.

Let us now suppose that \( M_n \) is an almost contact metric hypersurface with the structure \(- (F, T, A, g)\) of the Kahler manifold with the structure \(- (J, G)\). Then we have
\[
J(BX) = B\tilde{X} + A(X)M
\]
\[
J(M) = -BT
\]
\[
(D_xA)(Y) = -h(X, \tilde{Y})
\]
\[
(D_xF)(Y) = -h(X, Y)T + A(Y)HX.
\]

If we define
\[
'F(BX, BY) = G(JBX, BY)
\]
\[
'F(X, Y) = g.(\tilde{X}, Y),
\]
we have
\[
'F(BX, BY) = G(JBX, BY) = G(B\tilde{X}, BY) = g(\tilde{X}, Y) = 'F(X, Y).
\]

Theorem 2.1. In order that an almost contact metric hypersurface \( M_n \) in a Kahler manifold \( M_{n+1} \) be normal, it is necessary and sufficient that \( F \) and \( H \) commute.

Proof: Let us consider
\[
N(X, Y) + dA(X, Y)T
\]
\[
= (D_xF)(Y) - (D_yF)(X) + \overline{(D_yF)(X)} - \overline{(D_xF)(Y)}
\]
\[
+ \{(D_xA)(Y) - (D_yA)(X)\} T.
\]

Substituting (2.13) in the right hand side of (2.15 a), we get
\[
N(X, Y) + dA(X, Y)T = A(Y)\{H\tilde{X} - \overline{HX}\} - A(X)\{H\tilde{Y} - \overline{HY}\}
\]
If $M_n$ is normal then from (1.5) and (2.15b), we have

$$A(Y)\{H\bar{X} - \bar{H}X\} - A(X)\{H\bar{Y} - \bar{H}Y\} = 0,$$

from which

$$A(Y)\{h(\bar{X}, Z) + h(X, \bar{Z})\} - A(X)\{h(\bar{Y}, Z) + h(Y, \bar{Z})\} = 0$$

for arbitrary vector field $X, Y, Z$ and consequently

$$h(\bar{X}, Z) + h(X, \bar{Z}) = vA(X)A(Z)$$

$v$ being a certain function. By contraction we have from this equation that $v = 0$. Thus

$$h(\bar{X}, Y) + h(X, \bar{Y}) = 0$$

or

$$H\bar{X} - \bar{H}X = 0 \quad (2.15c)$$

which shows that $F$ and $H$ commute. Conversely if $F$ and $H$ commute then from (2.15b), we get (1.5). Hence $M_n$ is normal.

**Theorem 2.2.** In order that $F$ and $H$ commute it is necessary and sufficient that $T$ be a Killing vector or that the tensor field $h$ satisfies

$$h(X, Y) = A(X)A(Y)h(T, T) + h(\bar{X}, \bar{Y}). \quad (2.16)$$

**Proof:** Commutation of $F$ and $H$ yields

$$h(\bar{X}, Y) + h(X, \bar{Y}) = 0 \quad (2.17)$$

which by virtue of (2.12) gives

$$(D_xA)(Y) + (D_yA)(X) = 0.$$ 

Hence $T$ is a Killing vector. Conversely if $T$ is a Killing vector (2.17) holds. Barring $Y$ in (2.17) and using (1.1), we get

$$h(\bar{X}, \bar{Y}) - h(X, Y) + A(Y)h(X, T) = 0. \quad (2.18)$$

Interchanging $X$ and $Y$ in the above and subtracting the resulting expression we get

$$A(X)h(Y, T) - A(Y)h(T, X) = 0.$$
which by putting \( Y = T \) gives
\[
h(X, T) = A(X) h(T, T). \tag{2.19}
\]
Substituting (2.19) in (2.18) we get (2.16). It is easy to show that (2.16) gives (2.17).

**Remark:** \( M_n \) is minimal hypersurface of \( M_{n+1} \) if and only if \( h(T, T) = 0 \) because by contraction (2.16) we find that the trace of \( H \) is equal to \( h(T, T) \).

**Theorem 2.3.** A normal almost contact metric hypersurface \( M_n \) of constant sectional curvature in a Kahler manifold \( M_{n+1} \) of constant holomorphic sectional curvature is quasi-umbilical.

**Proof:** If a Kahler manifold is of constant holomorphic sectional curvature we have
\[
'R(BX, BY, BZ, BU) = -\frac{k}{4} [g(X, U)g(Y, Z) - g(Y, U)g(X, Z) + 'F(X, U) \times 'F(Y, Z) - 'F(Y, U) 'F(X, Z) - 2'F(X, Y) 'F(Z, U)]
\]
Putting this and (1.6) in Gauss characteristic equation (2.7), we get
\[
(k_2 - k_1) [g(X, U) g(Y, Z) - g(Y, U) g(X, Z)]
+ k_2 ['F(X, U) 'F(Y, Z) - 'F(Y, U) 'F(X, Z) - 2'F(X, Y) 'F(Z, U)]
= h(Y, U) h(X, Z) - h(X, U) h(Y, Z). \tag{2.20}
\]
If \( M_n \) is normal it will satisfy (2.19). Putting \( T \) for \( Y \) and \( Z \) in (2.20) and using the fact that \( 'F(X, T) = 0 \), we have
\[
(k_2 - k_1) [g(X, U) - A(X) A(U)]
= h(T, U) h(T, X) - h(T, T) h(X, U), \tag{2.21}
\]
which by virtue of (2.19) gives
\[
h(X, U) = a g(X, U) + \beta A(X) A(U)
\]
where
\[
a = \frac{k_1 - k_2}{h(T, T)} \quad \text{and} \quad \beta = h(T, T) - a.
Thus $M_n$ is quasi-umbilical.

**Corollary 2.1.** Under the hypothesis of the Theorem 2.3, the hypersurface $M_n$ is cylindrical if and only if $k_1 = k_2$.

**Theorem 2.4.** An umbilical almost contact metric hypersurface $M_n$ in a Kahler manifold $M_{n+1}$ is normal.

**Proof:** An almost contact metric hypersurface is normal if and only if (2.17) holds. From (1.3), we have

$$g(\tilde{X}, Y) + g(X, \tilde{Y}) = 0.$$  

If $M_n$ is umbilical above equation gives

$$h(\tilde{X}, Y) + h(X, \tilde{Y}) = 0$$

that is (2.17) holds. Thus we have the statement.

**Corollary 2.2.** In an umbilical almost contact metric hypersurface $M_n$ in a Kahler manifold $M_{n+1}$, we have

$$(D_x'F)(Y, T) - (D_y'F)(X, T) = 0.$$  

**Proof:** If $M_n$ is umbilical then from (2.13), we get

$$(D_xF)(Y) = -ag(X, Y)T + aA(Y)X.$$  

Interchanging $X$ and $Y$ and subtracting the result from above we get

$$(D_xF)(Y) - (D_yF)(X) = a\{A(Y)X - A(X)Y\}$$  

which gives

$$(D'F)(Y, Z) - (D'F)(X, Z) = a\{A(Y)g(X, Z) - A(X)g(Y, Z)\}.$$  

Putting $Z = T$, we have the result.

**Theorem 2.5.** An umbilical almost contact metric hypersurface, whose second fundamental form $h$ satisfies $h(T, T) = -1$ in a Kahler manifold, admits $'F$ as a conformal Killing tensor whose associated 1-form is $A$.

**Proof:** From (2.3) we see that $'\tilde{F}$ is covariant constant with respect to the Riemannian connexion $E$. Using this fact and (2.5) in the derivative of (2.14) with respect to $E$, we get

$$(D_x'F)(Y, Z) = h(X, Y)'\tilde{F}(M, BZ) - h(X, Z)'\tilde{F}(M, BY).$$  

(2.22)
Using (2.11) in the definition of $\tilde{\mathcal{F}}$, we see that
\[
\tilde{\mathcal{F}}(M, BX) = - A(X).
\]
Therefore (2.22) becomes
\[
(D_{X'}F)(Y, Z) = - h(X, Y) A(Z) + h(X, Z) A(Y).
\]

If the hypersurface is umbilical and $h(T, T) = -1$, (2.23) reduces to
\[
(D_{X'}F)(Y, Z) = g(X, Y) A(Z) - g(X, Z) A(Y).
\]
Interchanging $X$ and $Y$ in the above expression and adding the result to the above we get
\[
(D_{X'}F)(Y, Z) + (D_{Y'}F)(X, Z)
= 2A(Z) g(X, Y) - A(X) g(Y, Z) - A(Y) g(X, Z)
\]
which shows that $\mathcal{F}$ is a conformal Killing tensor.

REFERENCE