A NOTE ON MEANS OF ENTIRE FUNCTIONS

G. H. FRICKE

(Department of Mathematics, Wright State University, Dayton, Ohio, U.S.A.)

AND

S. M. SHAH, F.A.Sc.

(Department of Mathematics, University of Kentucky, Lexington, Kentucky, U.S.A.)

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ABSTRACT

A result of Singh and Sreenivasulu is proved under less restrictive hypothesis. It is also shown that if a condition on the coefficients is not satisfied, the theorem will not hold.

Let \( f(z) \) be an entire function. Denote the mean of order \( \delta, 0 < \delta < \infty \), by

\[
M_\delta(r, f) = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \right\}^{1/\delta}
\]

and the maximum modulus by \( M(r, f) \).

Singh and Sreenivasulu consider \( M_\delta(r, f)/M(r, f) \) and state the following theorem in [4].

**Theorem A.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function with \( a_n \) real.

Also let \( \delta > 0 \) be a constant and

\[
X(z) = \left\{ f(z) \right\}^{\delta/2} = \sum_{n=0}^{\infty} c_n z^n,
\]

where \( \delta_1 \) is the first even integer greater than \( \delta \). If

\[
R_n = \frac{a_{n-1}}{a_n} \quad \text{and} \quad R_n^1 = \frac{c_{n-1}}{c_n}
\]

are both strictly increasing and further if

\[
\limsup_{n \to \infty} \frac{a_n^2}{a_{n-1}a_{n+1}} = 1
\]
then
\[ \lim_{r \to 0} \frac{M_\delta(r, f)}{M(r, f)} = 0. \]

In this brief note we show that some of these conditions are unnecessary. We prove

**Proposition 1.** Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be an entire function. If, for some \( \alpha, \beta \) with \( \beta - \alpha < \pi \) and for \( n \) sufficiently large, \( \alpha \leq \arg a_n \leq \beta \) and if further
\[ \lim_{r \to 0} \frac{\mu(r, f)}{M(r, f)} = 0 \]

(where \( \mu(r, f) \) denotes the maximum term) then
\[ \lim_{r \to \infty} \frac{M_\delta(r, f)}{M(r, f)} = 0 \]
for all \( \delta > 0 \). If
\[ \lim_{r \to \infty} \frac{\mu(r, f)}{M(r, f)} = 0 \]
then
\[ \lim_{r \to \infty} \frac{M_\delta(r, f)}{M(r, f)} = 0. \]

**Proposition 2.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function with \( \alpha \leq \arg a_n \leq \beta \) for \( n > n_0 \) and \( \beta - \alpha < \pi \). If \( |a_n|/a_{n+1} \) is non-decreasing for \( n > n_1 \) and if \( \limsup_{n \to \infty} \frac{a_n^2/a_{n-1}a_{n+1}}{a_{n+1}^2/a_{n}a_{n+1}} = 1 \) then
\[ \lim_{r \to \infty} \frac{M_\delta(r, f)}{M(r, f)} = 0. \]

It is easy to show that for entire functions defined by gap power series satisfying certain conditions we have
\[ \lim_{r \to \infty} \frac{M_\delta(r, f)}{M(r, f)} = 1. \]

We will prove the following extension.

**Proposition 3.** Let \( 0 \leq \rho \leq \infty \). There exists an entire function \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) of order \( \rho \) with \( a_n > 0 \) and \( a_n|a_{n+1} \) non-decreasing for all \( n \) such that \( \limsup_{r \to \infty} \frac{M_\delta(r, f)}{M(r, f)} = 1. \)
Proof of Proposition 1.—We note that $M_\delta(r, f)$ increases with $\delta$ [2, p. 143]. Further we will show that if for some $\delta > 0$, $M_\delta(r, f) = o \{M(r, f)\}$ as $r \to \infty$, then this result holds for every $\delta > 0$.

Let $0 < a < 1$ and $E = \{\theta \| f(re^{i\theta}) \| > a M(r, f)\}$.

Clearly,

$$M_\delta(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{\delta} d\theta \right\}^{1/\delta} \geq aM(r, f) \left\{ \frac{m(E)}{2\pi} \right\}^{1/\delta}$$

Here $m(E)$ denotes the measure of $E$.

Now, if $M_\delta(r, f) = o \{M(r, f)\}$ for some $\delta$ then $m(E) \to 0$ as $r \to \infty$. Therefore, for $CE = [0, 2\pi] \setminus E$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\Delta d\theta$$

$$= \frac{1}{2\pi} \int_E |f(re^{i\theta})|^\Delta d\theta + \frac{1}{2\pi} \int_{CE} |f(re^{i\theta})|^\Delta d\theta$$

$$\leq M(r, f)^\Delta \left( \frac{m(E)}{2\pi} \right) + a^\Delta M(r, f)^\Delta \left( \frac{m(CE)}{2\pi} \right)$$

$$\leq M(r, f)^\Delta \left\{ \frac{m(E)}{2\pi} + a^\Delta \right\}.$$ 

Since $m(E) \to 0$ as $r \to \infty$ and $a$ can be chosen arbitrarily small we must have $M_\Delta(r, f) = o \{M(r, f)\}$ as $r \to \infty$.

So we need to prove the proposition only for $\delta = 2$. From the hypothesis we obtain for $r > r_0$ and some $A \geq 1$, $M(r, f) \geq (1/A) \sum_{n=0}^\infty |a_n| r^n$. Furthermore, let $\mu(r, f) = K(r) M(r, f)$, then $K(r) \to 0$ as $r \to \infty$.

Thus, for $r > r_0$,

$$\{M_2(r, f)\}^2 = \sum_{n=0}^\infty |a_n|^2 r^{2n} \leq \mu(r, f) \sum_{n=0}^\infty |a_n| r^n$$

$$\leq \mu(r, f) M(r, f)$$

$$= K(r) A \{M(r, f)\}^2.$$
Since $K(r) A \to 0$ as $r \to 0$ we have
\[
\lim_{r \to \infty} \frac{M_2(r, f)}{M(r, f)} = 0.
\]
Similarly the second part follows since \( \lim_{r \to \infty} K(r) = 0. \)

Remark.—If there is a complex number \( a \) such that \( f(z) \neq a \) for all \( z \), then Clunie and Hayman [1] have shown that \( \lim \inf_{r \to \infty} K(r) = 0. \)

Proof of Proposition 2.—S. M. Shah [3] showed that \( |a_n/a_{n+1}| \) strictly increasing and \( \lim \sup_{r \to \infty} |a_n^2/a_{n-1}a_{n+1}| = 1 \) imply \( \lim \inf_{r \to \infty} \mu(r, f)/M(r, f) = 0. \)

His argument [3: p. 422] works also if \( |a_n/a_{n+1}| \) is non-decreasing. The result now follows from Proposition 1. We may note that the condition \( a \leq \arg a_n \leq \beta \) can be relaxed as to allow for example \( a_n \) to be alternately positive and negative. The only necessary condition is that for some \( A \geq 1, \)
\[
M(r, f) \geq \frac{1}{A} \sum_{n=0}^{\infty} |a_n| r^n \quad \text{for all } r > r_0.
\]

Proof of Proposition 3.—Consider first the case \( 0 < \rho < \infty \) and let \( \{\lambda_n\}_{1}^{\infty} \) be a rapidly increasing sequence of positive integers. Define a sequence \( \{R_n\} \) by
\[
R_j = \lambda_j^{1/\rho} \text{ for } \lambda_{n-1} < j \leq \lambda_n.
\]
For ease of notation we sometimes write \( R_n = R(n) \). Then we have
\[
1 = R_1 = \ldots = R(\lambda_1 - 1), \quad R(\lambda_n) < R(\lambda_n + 1) = \ldots = R(\lambda_{n+1})
\]
Then, let
\[
f(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_1R_2 \ldots R_n}.
\]
Clearly
\[
\mu(r, f) = \frac{r^n}{R_1R_2 \ldots R_n} \quad \text{for } R_n \leq r < R_{n+1}.
\]
We now choose \( R \) such that
\[
R(\lambda_n) < R < R(\lambda_n + 1) = R(\lambda_{n+1}),
\]
\[
\frac{R}{R(\lambda_n)} \geq n \lambda_n \quad \text{ (1)}
\]
and
\[
\frac{R(\lambda_n + 1)}{R} \geq 2n. \tag{2}
\]

The existence of such an \( R \) is easily verified by the fact that \( \{\lambda_j\}_1^\infty \) is rapidly increasing.

Now, for \( 0 < \theta < 2\pi \),
\[
|f(Re^{i\theta})| \geq \mu(R, f) - \sum_{j=1}^{\lambda_n-1} \frac{R^j}{R_1 R_2 \ldots R_j} - \sum_{j=\lambda_n+1}^{\infty} \frac{R^j}{R_1 R_2 \ldots R_j}
\]
\[
= \mu(R, f) - \Sigma_1 - \Sigma_2.
\]

For \( j < \lambda_n \) we have, since \( R > R_j \),
\[
\frac{R^j}{R_1 R_2 \ldots R_j} \geq \frac{R^{j-1}}{R_1 \ldots R_{j-1}}.
\]

Thus,
\[
\sum_1^j \leq \frac{\lambda_n}{\lambda_n R_1 R_2 \ldots R(\lambda_n - 1)} \frac{R^{\lambda_n - 1}}{R_1 R_2 \ldots R(\lambda_n - 1)}
\]
\[
\leq \frac{R}{nR(\lambda_n) R_1 R_2 \ldots R(\lambda_n - 1)} \frac{R^{\lambda_n}}{n R_1 \ldots R(\lambda_n)} = \frac{\mu(R, f)}{n}.
\]

Also, since
\[
\frac{R}{R_j} \leq \frac{1}{2n} \quad \text{for} \quad j > \lambda_n \quad \text{by (2)},
\]

we have
\[
\frac{R^j}{R_1 R_2 \ldots R_j} \leq \left(\frac{1}{2n}\right)^{j-\lambda_n} \frac{R^{\lambda_n}}{R_1 \ldots R(\lambda_n)} \quad \text{for} \quad j > \lambda_n.
\]
Hence,
\[ \sum_{2}^{n} \leq \frac{R^{\lambda_{n}}}{R_{1} \ldots R(\lambda_{n})} \sum_{j=1}^{\infty} \left( \frac{1}{2n} \right)^{j} \]
\[ \leq \frac{\mu(R, f)}{n}. \]

Therefore, for \( 0 \leq \theta \leq 2\pi \)
\[ |f(Re^{i\theta})| \geq \left( 1 - \frac{2}{n} \right) \mu(R, f) \]
and similarly,
\[ |f(Re^{i\theta})| \leq \mu(R, f) + \Sigma_{1} + \Sigma_{2} \]
\[ \leq \left( 1 + \frac{2}{n} \right) \mu(R, f). \]

Thus,
\[ \lim_{r \to \infty} \sup_{r} \frac{M_{\delta}(r, f)}{M(r, f)} = 1. \]

Further \( a_{n} = 1/R_{1}R_{2} \ldots R_{n} > 0 \) and since \( \{\lambda_{n}\}_{1}^{\infty} \) is rapidly increasing \( \lambda_{n-1} = o(\lambda_{n}) \) as \( n \to \infty \).

Thus,
\[ \log \{R_{1}R_{2} \ldots R(\lambda_{n})\} \geq \log \{R(\lambda_{n-1} + 1) \ldots R(\lambda_{n})\} \]
\[ = (\lambda_{n} - \lambda_{n-1}) \log R(\lambda_{n}) = \{1 - o(1)\} \lambda_{n} \log R(\lambda_{n}) \text{ as } n \to \infty. \]

Therefore,
\[ \lim_{n \to \infty} \sup \frac{n \log n}{\log \frac{1}{a_{n}}} = \lim_{n \to \infty} \sup \frac{n \log n}{\log R_{1}R_{2} \ldots R_{n}} \]
\[ = \lim_{n \to \infty} \sup \frac{\lambda_{n} \log \lambda_{n}}{\log R_{1}R_{2} \ldots R(\lambda_{n})} \]
\[ = \lim_{n \to \infty} \sup \frac{\lambda_{n} \log \lambda_{n}}{\lambda_{n} \log R_{\lambda_{n}}} = \rho, \text{ i.e.,} \]
\( f(z) \) is of order \( \rho \).
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Easy modification in the definition of $R_n$ gives functions for which $\rho = 0$ and functions for which $\rho = \infty$. We omit these details.

REFERENCES

1. Clunie, J. and Hayman, W. K.

2. Hardy, G. H., Littlewood, J. E. and Polya, G.

3. Shah, S. M.

4. Singh, S. K. and Sreenivasulu