ON MHD FLOW ALONG AN INFINITE FLAT WALL WITH CONSTANT SUCTION

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Abstract

Exact solutions of the Navier-Stokes equations are derived by a Laplace-transform technique for two-dimensional, incompressible flow of an electrically conducting fluid past an infinite porous plate under the action of a transverse magnetic field subject to the conditions: (i) the magnetic Prandtl number P_m is unity, and (ii) the Alfven velocity is less than the suction velocity. It is assumed that the flow is independent of the distance parallel to the plate and that the velocity component normal to the plate is constant. General formulae are derived for the velocity distribution and the magnetic field in terms of the given external velocity. The skin-friction is obtained and some special cases are considered.

1. Introduction

Recently exact solutions have been obtained by Ramana Rao and Sithapathi[1] for the flow of an incompressible weakly conducting fluid past an infinite flat plate under the action of a transverse magnetic field, when the suction velocity normal to the plate is constant and the external flow velocity is a general function of time. This case corresponds to the magnetic Prandtl number $P_m \simeq 0$. In this paper solutions are obtained corresponding to $P_m = 1$, when the Alfven velocity is less than the suction velocity as in Soundalgekar.[2]

2. General Theory

Under the conditions mentioned above, it is easily seen that the governing equations of motion in non-dimensional form as in Soundalgekar are:

$$\frac{1}{4}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = \frac{1}{4}\frac{dU}{dt} + \frac{\partial^2 u}{\partial y^2} + k^2 \frac{\partial H}{\partial y} \tag{2.1}$$

$$\frac{1}{4}\frac{\partial H}{\partial t} - \frac{\partial H}{\partial y} - \frac{\partial u}{\partial y} = \frac{\partial^2 H}{\partial y^2} \tag{2.2}$$

which are to be solved subject to the boundary conditions

$$u = 0, \quad H = 0 \quad \text{at} \quad y = 0$$
$$u \rightarrow U, \quad H = 0 \quad \text{as} \quad y \rightarrow \infty. \tag{2.3}$$

Making use of the substitutions as in Sutton and Sherman,[3]

$$\xi = u + kH, \qquad \chi = u - kH \tag{2.4}$$

where k is the magnetic field parameter, the equations (2.1, 2.2) are uncoupled into two differential equations in ξ and χ.

$$\frac{1}{4}\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial y} = \frac{1}{4}\frac{dU}{dt} + \frac{\partial^2 \xi}{\partial y^2} + k\frac{\partial \xi}{\partial y} \tag{2.5}$$

$$\frac{1}{4}\frac{\partial \chi}{\partial t} - \frac{\partial \chi}{\partial y} = \frac{1}{4}\frac{dU}{dt} + \frac{\partial^2 \chi}{\partial y^2} - k\frac{\partial \chi}{\partial y} \tag{2.6}$$

and the boundary conditions (2.3) become

$$\xi = 0, \quad \chi = 0 \quad \text{at} \quad y = 0$$
$$\xi \rightarrow U, \quad \chi \rightarrow U \quad \text{as} \quad y \rightarrow \infty. \tag{2.7}$$

Clearly then the solution for χ is the same as for ξ except for a change in the sign of k, the magnetic field parameter. Accordingly the solution for ξ yields the solution of the problem in exact form.

An exact solution of (2.5) is of the form

$$U = 1 + f(t)$$
$$\xi = \zeta_0(y) + g(y, t) \tag{2.8}$$

where

$$\zeta_0(y) = 1 - \exp.[-(1 + k)y] \tag{2.9}$$

and $f(t)$ is a given arbitrary function and $g(y, t)$ is a function to be determined.

Substituting (2.8) in (2.5) and utilizing the fact that ζ_0 is given by (2.9) we obtain:

$$\frac{1}{4}\frac{\partial g}{\partial t} - \frac{\partial g}{\partial y} = \frac{1}{4}f'(t) + \frac{\partial^2 g}{\partial y^2} + k\frac{\partial g}{\partial y} \tag{2.10}$$

with the boundary conditions that $g = 0$ at $y = 0$ and $g \rightarrow f(t)$ as $y \rightarrow \infty$.

Taking the Laplace transform of (2.10), we get

$$\frac{d^2\bar{g}}{dy^2} + (1+k)\frac{d\bar{g}}{dy} - \frac{s}{4}\bar{g} = -\frac{s}{4}\bar{f} \qquad (2.11)$$

with the boundary conditions

$$\bar{g} = 0 \quad \text{at} \quad y = 0$$

$$\bar{g} \to \bar{f} \quad \text{as} \quad y \to \infty. \qquad (2.12)$$

The solution of (2.11) subject to (2.12) is

$$\bar{g} = \bar{f}\,[1 - \exp.(-\alpha y)], \qquad (2.13)$$

where

$$\alpha = \tfrac{1}{2}\,[1 + k + \sqrt{(1+k)^2 + s}]. \qquad (2.14)$$

On inverting (2.13), we obtain

$$g(y,t) = f(t) - \frac{y}{4\sqrt{\pi}}\exp.\left[-\frac{1}{2}(1+k)\,y\right]\int_0^t \lambda^{-(3/2)} f(t-\lambda) \times \exp.\left[-(1+k)^2\lambda - \frac{y^2}{16\lambda}\right]d\lambda. \qquad (2.15)$$

Substituting (2.9) and (2.15) in (2.8), we obtain

$$\xi = 1 - \exp.[-(1+k)\,y] + f(t) - \frac{y}{4\sqrt{\pi}}\exp.\left[-\frac{1}{2}(1+k)\,y\right] \times \int_0^t \lambda^{-(3/2)} f(t-\lambda)\exp.\left[-(1+k)^2\lambda - \frac{y^2}{16\lambda}\right]d\lambda. \qquad (2.16)$$

Noting the expression for χ is the same as for ξ except for a change in the sign of k, we obtain:

$$\chi = 1 - \exp.[-(1-k)\,y] + f(t) - \frac{y}{4\sqrt{\pi}}\exp.\left[-\frac{1}{2}(1-k)\,y\right] \times \int_0^t \lambda^{-(3/2)} f(t-\lambda)\exp.\left[-(1-k)^2\lambda - \frac{y^2}{16\lambda}\right]d\lambda, \qquad (2.17)$$

where $k < 1$, otherwise the boundary condition on χ at infinity will be violated and this implies that the Alfven velocity should be less than the suction velocity as mentioned earlier.

In view of (2.4), we obtain

$$u = 1 + f(t) - \exp.(-y)\cosh(ky) - \frac{y}{8\sqrt{\pi}}\exp.(-y/2)$$

$$\times \int_0^t \lambda^{-(3/2)} f(t-\lambda)\exp.\left(-\frac{y^2}{16\lambda}\right)\left\{\exp.\left[-\frac{ky}{2}-(1+k)^2\lambda\right]\right.$$

$$\left.+\exp.\left[\frac{ky}{2}-(1-k)^2\lambda\right]\right\}d\lambda, \tag{2.18}$$

$$H = \frac{1}{k}\left[\exp.(-y)\sinh(ky) - \frac{y}{8\sqrt{\pi}}\exp.\left(-\frac{y}{2}\right)\int_0^t \lambda^{-(3/2)} f(t-\lambda)\right.$$

$$\times\exp.\left(-\frac{y^2}{16\lambda}\right)\left\{\exp.\left(-\frac{ky}{2}-(1+k)^2\lambda\right)\right.$$

$$\left.\left.-\exp.\left(\frac{ky}{2}-(1-k)^2\lambda\right)\right\}d\lambda\right]. \tag{2.19}$$

The skin-friction, in non-dimensional form, is given by

$$\tau = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 1 + \frac{1}{2}f(t) - \frac{1}{8\sqrt{\pi}}\int_0^t \lambda^{-(3/2)} f(t-\lambda)$$

$$\times\left[\exp.\{-(1+k)^2\lambda\} + \exp.\{-(1-k)^2\lambda\}\right]d\lambda. \tag{2.20}$$

3. Applications

3.1. *Impulsive velocity field*—Let $f(t) = \triangle H(t)$, where $\triangle$ is constant and $H(t)$ is the unit function. The velocity given by (2.18) is

$$u = 1 + \triangle H(t) - \exp.(-y)\cosh(ky)$$

$$-\frac{\triangle H(t)}{4}\left[\exp.\{-(1+k)y\}\,\mathrm{erfc}\left(\frac{y}{4\sqrt{t}}-(1+k)\sqrt{t}\right)\right.$$

$$+\,\mathrm{erfc}\left(\frac{y}{4\sqrt{t}}+(1+k)\sqrt{t}\right)+\exp.\{-(1-k)y\}$$

$$\left.\times\,\mathrm{erfc}\left(\frac{y}{4\sqrt{t}}-(1-k)\sqrt{t}\right)+\mathrm{erfc}\left(\frac{y}{4\sqrt{t}}+(1-k)\sqrt{t}\right)\right] \tag{3.1}$$

for $t \neq 0$. It is easily seen from (3.1) that as $t \to \infty$

$$u \to (1 + \triangle)[1 - \exp.(-y)\cosh(ky)]. \tag{3.2}$$

The magnetic field given by (2.19) is

$$H = \frac{1}{k}\Big[\exp.(-y)\sinh(ky) - \frac{\triangle H(t)}{4}\Big\{\exp.\big(-(1+k)y\big)$$

$$\times \operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} - (1+k)\sqrt{t}\Big) + \operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} + (1+k)\sqrt{t}\Big)$$

$$- \exp.\big(-(1-k)y\big)\operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} - (1-k)\sqrt{t}\Big)$$

$$- \operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} + (1-k)\sqrt{t}\Big)\Big\}\Big] \tag{3.3}$$

for $t \neq 0$. As $t \to \infty$, (3·3) becomes

$$H \to \frac{1}{k}(1 + \triangle)\exp.(-y)\sinh(ky). \tag{3.4}$$

The skin-friction τ given by (2.20) is for $t \neq 0$,

$$\tau = 1 + \frac{\triangle H(t)}{2} + \frac{\triangle H(t)}{4}\Big[\frac{1}{\sqrt{t\pi}}\exp.\{-(1+k)^2 t\} + (1+k)$$

$$\times \operatorname{erf}\{(1+k)\sqrt{t}\} + \frac{1}{\sqrt{t\pi}}\exp.\{-(1-k)^2 t\}$$

$$+ (1-k)\operatorname{erf}\{(1-k)\sqrt{t}\}\Big]. \tag{3.5}$$

As $t \to \infty$, (3.5) becomes

$$\tau = 1 + \triangle. \tag{3.6}$$

3.2. *Accelerated velocity field.*—Let $f(t) = \triangle t H(t)$ where $H(t)$ is the unit function. Equation (2.18) yields

$$u = 1 + \triangle t H(t) - \exp.(-y)\cosh(ky)$$

$$- \frac{\triangle H(t)}{4}\Big[t\Big\{\exp.\big(-y(1+k)\big)\operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} - (1+k)\sqrt{t}\Big)$$

$$+ \operatorname{erfc}\Big(\frac{y}{4\sqrt{t}} + (1+k)\sqrt{t}\Big) + \exp.\big(-y(1-k)\big)$$

$$\times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1-k)\sqrt{t}\right)+\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1-k)\sqrt{t}\right)\Big\}\Big]$$

$$+\frac{y\triangle \mathrm{H}(t)}{16}\Big[\frac{1}{1+k}\Big\{\exp.(-y(1+k))$$

$$\times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1+k)\sqrt{t}\right)-\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1+k)\sqrt{t}\right)\Big\}$$

$$+\frac{1}{1-k}\Big\{\exp.(-y(1-k))\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1-k)\sqrt{t}\right)$$

$$-\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1-k)\sqrt{t}\right)\Big\}\Big] \tag{3.7}$$

for $t\neq 0$. As $t\to\infty$ equation (3.7) becomes

$$u\to(1+\triangle t)\left[1-\exp.(-y)\cosh(ky)\right]$$

$$+\frac{\triangle y}{8}\exp.(-y)\left[\frac{1}{1+k}\exp.(-ky)+\frac{1}{1-k}\exp.(ky)\right]. \tag{3.8}$$

The magnetic field given by (2.19) is

$$\mathrm{H}=\frac{1}{k}\Big[\exp.(-y)\sinh(ky)-\frac{\triangle t\mathrm{H}(t)}{4}\Big\{\exp.(-y(1+k))$$

$$\times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1+k)\sqrt{t}\right)+\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1+k)\sqrt{t}\right)$$

$$-\exp.(-y(1-k))\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1-k)\sqrt{t}\right)$$

$$-\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1-k)\sqrt{t}\right)\Big\}+\frac{y\triangle \mathrm{H}(t)}{16}$$

$$\times\Big\{\frac{1}{1+k}\Big[\exp.(-y(1+k))\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1+k)\sqrt{t}\right)$$

$$-\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1+k)\sqrt{t}\right)\Big)\Big]-\frac{1}{1-k}$$

$$\times \Big[\exp.\left(-y(1-k)\right)\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}-(1-k)\sqrt{t}\right)$$

$$-\operatorname{erfc}\left(\frac{y}{4\sqrt{t}}+(1-k)\sqrt{t}\right)\Big]\Big\}\Big]. \quad (3.9)$$

for $t \neq 0$. As $t \to \infty$ equation (3.9) yields

$$H \to \frac{1}{k}\Big[(1+\triangle t)\exp.(-y)\sinh(ky)+\frac{\triangle y}{8}\exp.(-y)$$

$$\times\left\{\frac{1}{1+k}\exp.(-ky)-\frac{1}{1-k}\exp.(ky)\right\}\Big]. \quad (3.10)$$

The skin-friction τ given by (2.20) is for $t \neq 0$,

$$\tau = 1+\frac{\triangle t H(t)}{2}+\frac{\triangle H(t)}{4}\Big[\sqrt{\frac{t}{\pi}}\{\exp.\left(-(1+k)^2 t\right)$$

$$+\exp.\left(-(1-k)^2 t\right)\}+t\{(1+k)\operatorname{erf}\left((1+k)\sqrt{t}\right)$$

$$+(1-k)\operatorname{erf}\left((1-k)\sqrt{t}\right)\}+\frac{1}{2}\left\{\frac{1}{1+k}\operatorname{erf}\left((1+k)\sqrt{t}\right)\right.$$

$$\left.+\frac{1}{1-k}\operatorname{erf}\left((1-k)\sqrt{t}\right)\right\}\Big]. \quad (3.11)$$

As $t \to \infty$ (3.11) gives

$$\tau \to 1+\triangle\left[t+\frac{1}{4(1-k^2)}\right]. \quad (3.12)$$

3.3. *Fluctuating velocity field.*—Let $f(t) = \epsilon \exp.(i\omega t)$ where

$$\omega = 4\frac{\nu\omega'}{V_0{}^2}.$$

Then equation (2.18) yields

$$u = 1+\epsilon\exp.(i\omega t)-\exp.(-y)\cosh(ky)-\frac{\epsilon\exp.(i\omega t)}{4}$$

$$\times\Big[\exp.\left\{-\frac{y}{2}\left(1+k-\sqrt{i\omega+(1+k)^2}\right)\right\}\Big\{\exp.\left(-y\sqrt{i\omega+(1+k)^2}\right)$$

$$\times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{t\left(i\omega + (1+k)^2\right)}\right) + \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{t\left(i\omega + (1+k)^2\right)}\right)\Bigg\}$$

$$+ \exp.\left\{-\frac{y}{2}\left(1 - k - \sqrt{i\omega + (1-k)^2}\right)\right\}\left\{\exp.\left(-y\sqrt{i\omega + (1-k)^2}\right)\right.$$

$$\times \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{t\left(i\omega + (1-k)^2\right)}\right) + \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{t\left(i\omega + (1-k)^2\right)}\right)\Bigg\}\Bigg] \tag{3.13}$$

for $t \neq 0$. As $t \to \infty$, equation (3.13) yields

$$u \to 1 - \exp.(-y)\cosh(ky) + \epsilon \exp.(i\omega t)$$

$$\times \left[1 - \frac{1}{2}\left\{\exp.\left(-\frac{y}{2}\left(1 + k + \sqrt{i\omega + (1+k)^2}\right)\right)\right.\right.$$

$$\left.\left. + \exp.\left(-\frac{y}{2}\left(1 - k + \sqrt{i\omega + (1-k)^2}\right)\right)\right\}\right] \tag{3.14}$$

in agreement with Soundalgekar's work for $A = 0$.

The magnetic field given by (2.19) is for $t \neq 0$,

$$H = \frac{1}{k}\Bigg[\exp.(-y)\sinh(ky) - \frac{\epsilon \exp.(i\omega t)}{4}\left\{\exp.\left(-\frac{y}{2}\left(1 + k - \sqrt{i\omega + (1+k)^2}\right)\right)\right.$$

$$\times \left[\exp.\left(-y\sqrt{i\omega + (1+k)^2}\right)\operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{t\left(i\omega + (1+k)^2\right)}\right)\right.$$

$$\left. + \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{t\left(i\omega + (1+k)^2\right)}\right)\right] - \exp.\left(-\frac{y}{2}\left(1 - k - \sqrt{i\omega + (1-k)^2}\right)\right)$$

$$\times \left[\exp.\left(-y\sqrt{i\omega + (1-k)^2}\right)\operatorname{erfc}\left(\frac{y}{4\sqrt{t}} - \sqrt{t\left(i\omega + (1-k)^2\right)}\right)\right.$$

$$\left.\left. + \operatorname{erfc}\left(\frac{y}{4\sqrt{t}} + \sqrt{t\left(i\omega + (1-k)^2\right)}\right)\right]\right\}\Bigg]. \tag{3.15}$$

As $t \to \infty$ equation (3.15) yields,

$$H \to \frac{1}{k}\left[\exp.(-y)\sinh(ky) + \frac{\epsilon}{2}\exp.(i\omega t)\right.$$

$$\times \left\{\exp.\left(-\frac{y}{2}\left(1 - k + \sqrt{i\omega + (1-k)^2}\right)\right)\right.$$

$$\left.\left. - \exp.\left(-\frac{y}{2}\left(1 + k + \sqrt{i\omega + (1+k)^2}\right)\right)\right\}\right] \tag{3.16}$$

in agreement with Soundalgekar's work for $A = 0$.

The skin-friction τ given by (2.20) is for $t \neq 0$,

$$\tau = 1 + \frac{\epsilon \exp. (i\omega t)}{2} + \frac{\epsilon \exp. (i\omega t)}{4} \left[\frac{1}{\sqrt{t\pi}} \{\exp. [-t(i\omega + (1 + k)^2)] \right.$$

$$+ \exp. [-t(i\omega + (1 - k)^2)]\} + \sqrt{i\omega + (1 + k)^2}$$

$$\times \operatorname{erf} [\sqrt{t(i\omega + (1 + k)^2)}] + \sqrt{i\omega + (1 - k)^2}$$

$$\left. \times \operatorname{erf} [\sqrt{t(i\omega + (1 - k)^2)}] \right]. \quad (3.17)$$

As $t \to \infty$ equation (3.17) yields

$$\tau \to 1 + \frac{\epsilon \exp. (i\omega t)}{4} [2 + \sqrt{i\omega + (1 + k)^2} + \sqrt{i\omega + (1 - k)^2}] \quad (3.18)$$

in agreement with equation (29) of Soundalgekar for A = 0.

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