1. Introduction

In this paper all modules are considered with reference to a base ring $R$. Further $R$ is assumed to be commutative and to possess an identity element. In other words, all modules are assumed to be commutative and unital. The identity mapping is denoted by $i$ without any subscripts. The notation and terminology will be as in [4].

This paper is a continuation of [2], where we have defined a sequence $A \to B \to C$ to be a semi-split sequence iff it is semi-exact and $\text{Ker}(BC)$ is a direct summand of $B$. Towards the end of [2] the following theorem has been proved:

**Theorem A**: Suppose

$$O \to A \to B \to C \to O$$

semi-splits. Then

$$O \to T(A) \to T(B) \to T(C) \to O$$

semi-splits if $T$ is an exact covariant functor and

$$O \to T(C) \to T(B) \to T(A) \to O$$

semi-splits if $T$ is an exact contravariant functor.

In section 2, Theorem A has been used to prove some key results which are necessary and sufficient conditions for short sequences to semi-split.

In section 3, some results about projective, injective and flat modules are proved by two methods—firstly by using semi-split sequences and secondly by using exact sequences. These show that semi-split sequences can be used as an alternative for exact sequences on certain subcategories of the category of all modules.

In section 4, a particular case when $R$ is Noetherian is considered.
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The base ring \( R \) is assumed to be commutative due to the following reasons:

\[(a) \ \text{Hom}_R(A, \ B) \text{ and } A \otimes_R B \text{ will be modules.} \]
\[(b) \ A \otimes_R B \text{ and } B \otimes_R A \text{ will be isomorphic as modules.} \]
\[(c) \text{ Every module over } R \text{ can be considered as a bimodule over } R. \]

Further the following well-known module isomorphisms are freely made use of:

\[
\text{Hom}_R(A, \text{Hom}_R(B, C)) \approx \text{Hom}_R(A \otimes_R B, C) \approx \text{Hom}_R(B \otimes_R A, C)
\]
\[
\text{Hom}_R(B, \text{Hom}_R(A, C))
\]

for all modules \( A, \ B, \ C. \)

Also when \( F \) is a free module with a finite base,

\[
F \otimes_R \text{Hom}_R(A, B) \approx \text{Hom}_R(\text{Hom}_R(F, A), B)
\]

for all modules \( A, \ B. \)

2. We now prove a few results which are necessary and sufficient conditions for a short sequence to semi-split.

\textbf{Theorem B:} A sequence

\[
\begin{array}{cccc}
  f & g \\
O & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & O
\end{array}
\]

semi-splits iff the sequence

\[
\begin{array}{cccc}
  & & X \otimes_R A & \rightarrow & X \otimes_R B & \rightarrow & X \otimes_R C & \rightarrow & O
\end{array}
\]

semi-splits, where \( X \) is flat.

\textit{Proof:} (a) Follows from Theorem A.

(b) Suppose

\[
\begin{array}{cccc}
  & i \otimes f & \rightarrow & i \otimes g \\
O & \rightarrow & X \otimes_R A & \rightarrow & X \otimes_R B & \rightarrow & X \otimes_R C & \rightarrow & O
\end{array}
\]

semi-splits, where \( X \) is flat. Then,

\[(i \otimes g)(i \otimes f) = 0 = i \otimes gf = 0 = gf = 0.\]

\[
X \otimes_R B = \text{Ker} (i \otimes g) \oplus X \otimes_R B'
\]
for every flat module $X$.

\[ S \Rightarrow X \otimes_R B = X \otimes_R \text{Ker}(g) \oplus X \otimes_R B' \]

\[ = X \otimes_R (\text{Ker}(g) \oplus B'), \]

Similarly,

\[ A = \text{Ker}(f) \oplus A' . \]

\[ \therefore O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \text{ semi-splits.} \]

**Corollary:** A sequence $O \rightarrow A \rightarrow B \rightarrow C \rightarrow O$ semi-splits iff the sequence $O \rightarrow A \otimes_R Y \rightarrow B \otimes_R Y \rightarrow C \otimes_R Y \rightarrow O$ semi-splits, where $Y$ is flat.

**Theorem C:** Suppose $P$ is projective. Then a sequence

\[ f \rightarrow g \]

\[ O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \]

semi-splits iff the sequence

\[ O \rightarrow \text{Hom}_R(P, A) \rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C) \rightarrow O \]

semi-splits.

**Proof:** (a) Follows from Theorem A.

(b) Suppose

\[ \text{Hom}(i, f) \rightarrow \text{Hom}_R(P, A) \rightarrow \text{Hom}_R(P, B) \]

\[ \text{Hom}(i, g) \rightarrow \text{Hom}_R(P, C) \rightarrow O \]

semi-splits, where $P$ is projective. Then,

\[ \text{Hom}(i, g) \text{Hom}(i, f) = O \Rightarrow \text{Hom}(i, gf) = O \Rightarrow gf = O \]

\[ \therefore \text{Hom}_R(P, B) = \text{Ker} \left( \text{Hom}(i, g) \right) \oplus \text{Hom}_R(P, B') \]

\[ = \text{Hom}_R(P, \text{Ker}(g)) \oplus \text{Hom}_R(P, B') \]

\[ = \text{Hom}_R(P, \text{Ker}(g) \oplus B'), \]
for all projective modules P.

\[ B = \text{Ker}(g) \oplus B'. \]

Similarly,

\[ A = \text{Ker}(f) \oplus A'. \]

\[ O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \]

semi-splits.

**Note:**

\[ \text{Ker} \left( \text{Hom}(i, g) \right) = \{ h \in \text{Hom}_R(P, B) / \text{Hom}(i, g) h = 0 \} \]
\[ = \{ h \in \text{Hom}_R(P, B) / gh = 0 \} \]
\[ = \{ h \in \text{Hom}_R(P, B) / h = f k, \text{ where } k \in \text{Hom}_R(P, A) \} \]

since P is projective

\[ = \text{Hom}_R(P, \text{Ker}(g)). \]

**Theorem D.**—Suppose Q is injective. Then a sequence

\[ f \quad g \]
\[ O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \]

semi-splits iff the sequence

\[ O \rightarrow \text{Hom}_R(C, Q) \rightarrow \text{Hom}_R(B, Q) \rightarrow \text{Hom}_R(A, Q) \rightarrow O \]

semi-splits.

**Proof.**—(a) Follows from Theorem A.

(b) Suppose \[ O \rightarrow \text{Hom}_R(C, Q) \rightarrow \text{Hom}_R(B, Q) \rightarrow \text{Hom}_R(A, Q) \rightarrow O \]

semi-splits, where Q is injective. Then,

\[ \text{Hom}(f, i) \text{Hom}(g, i) = 0 \Rightarrow \text{Hom}(gf, i) = 0 \Rightarrow gf = 0. \]

\[ \therefore \text{Hom}_R(B, Q) = \text{Ker} \left( \text{Hom}(f, i) \right) \oplus \text{Hom}_R(B', Q) \]
\[ = \text{Hom}_R(\text{Ker}(g), Q) \oplus \text{Hom}_R(B', Q) \]
\[ = \text{Hom}_R(\text{Ker}(g) \oplus \text{B'}, Q) \],

for all injective modules Q.

\[ \therefore B = \text{Ker}(g) \oplus B'. \]
Similarly, \( A = \text{Ker} (f) \oplus A' \).
\[
\therefore \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow O
\]
semi-splits.

**Note:**
\[
\text{Ker} (\text{Hom} (f, i)) = \{ h \in \text{Hom}_R (B, Q) / \text{Hom} (f, i) h = 0 \}
\]
\[
= \{ h \in \text{Hom}_R (B, Q) / hf = 0 \}
\]
\[
= \{ h \in \text{Hom}_R (B, Q) / h = kg, \}
\]
where
\[
k \in \text{Hom}_R (C, Q)
\]
since \( Q \) is injective
\[
= \text{Hom}_R (\text{Ker} (g), Q).
\]

3. In what follows, some results about projective, injective and flat modules are proved in two different ways.

**Theorem 3.1.** Suppose \( \text{Hom}_R (A, B) \) is injective. Then \( A \) is projective iff \( B \) is injective.

**First proof:** Suppose \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) semi-splits. Then, \( \text{Hom}_R (A, B) \) is injective \( \Leftrightarrow \) \( 0 \rightarrow \text{Hom}_R (C'', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C, \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C', \text{Hom}_R (A, B)) \rightarrow 0 \) semi-splits
\[
\therefore \quad A \text{ is projective iff } B \text{ is injective.}
\]

**Second proof:** \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) is exact and \( \text{Hom}_R (A, B) \) is injective \( \Leftrightarrow \) \( 0 \rightarrow \text{Hom}_R (C'', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C, \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C'', B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C', B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C, B)) \rightarrow 0 \) is exact.
\[
\therefore \quad A \text{ is projective iff } B \text{ is injective.}
\]

**Corollary 3.2.** If \( A \) is projective and \( B \) is injective, then \( \text{Hom}_R (A, B) \) is injective.

**Theorem 3.3.** Suppose \( \text{Hom}_R (A, B) \) is injective. Then \( A \) is flat iff \( B \) is injective.

**First proof:** Suppose \( 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \) semi-splits. Then, \( \text{Hom}_R (A, B) \) is injective \( \Leftrightarrow \) \( 0 \rightarrow \text{Hom}_R (C'', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C, \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C'', B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C', B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C, B)) \rightarrow \text{Hom}_R (A, \text{Hom}_R (C', B)) \rightarrow 0 \) is exact.
\[ \text{Hom}_R (C, \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C', \text{Hom}_R (A, B)) \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle O \rightarrow \text{Hom}_R (A \otimes_R C'', B) \rightarrow \text{Hom}_R (A \otimes_R C, B) \rightarrow \text{Hom}_R (A \otimes_R C', B) \rightarrow O \text{ semi-splits}. \]

\[ \therefore \text{A is flat iff B is injective.} \]

**Second proof.** \[ O \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow O \text{ is exact and } \text{Hom}_R (A, B) \text{ is injective} \]

\[ \langle = \rangle O \rightarrow \text{Hom}_R (C'', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C, \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (C', \text{Hom}_R (A, B)) \rightarrow \text{Hom}_R (A \otimes_R C'', B) \rightarrow \text{Hom}_R (A \otimes_R C, B) \rightarrow \text{Hom}_R (A \otimes_R C', B) \rightarrow O \text{ is exact} \]

\[ \langle = \rangle O \rightarrow \text{Hom}_R (A \otimes_R C'', B) \rightarrow \text{Hom}_R (A \otimes_R C, B) \rightarrow \text{Hom}_R (A \otimes_R C', B) \rightarrow O \text{ is exact}. \]

\[ \therefore \text{A is flat iff B is injective.} \]

**Corollary 3.4.** If A is flat and B is injective, then \( \text{Hom}_R (A, B) \) is injective.

**Theorem 3.5.** \( P' \otimes_R P'' \equiv \text{P''} \otimes_R P' \) is projective iff \( P' \) and \( P'' \) are projective and nonzero.

**First proof.** Suppose \( O \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow O \text{ semi-splits}. \) Then, \( P' \) and \( P'' \) are projective \( \langle = \rangle O \rightarrow \text{Hom}_R (P', \text{Hom}_R (P'', C)) \rightarrow \text{Hom}_R (P', \text{Hom}_R (P'', C)) \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle O \rightarrow \text{Hom}_R (P' \otimes_R P'', C') \rightarrow \text{Hom}_R (P' \otimes_R P'', C) \rightarrow \text{Hom}_R (P' \otimes_R P'', C'') \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle P' \otimes_R P'' \text{ is projective.} \]

**Second proof.** \( O \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow O \text{ is exact and } P' \) and \( P'' \) are projective \( \langle = \rangle O \rightarrow \text{Hom}_R (P', \text{Hom}_R (P'', C)) \rightarrow \text{Hom}_R (P', \text{Hom}_R (P'', C)) \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle O \rightarrow \text{Hom}_R (P' \otimes_R P'', C') \rightarrow \text{Hom}_R (P' \otimes_R P'', C) \rightarrow \text{Hom}_R (P' \otimes_R P'', C'') \rightarrow O \text{ is exact} \]

\[ \langle = \rangle P' \otimes_R P'' \text{ is projective.} \]

**Corollary 3.6.** \( P_{j_1} \otimes_R P_{j_2} \otimes_R \ldots \otimes_R P_{j_n} \) (where \( j_1, j_2, \ldots j_n \) is a permutation of \( 1, 2, \ldots n \)) will be projective iff each of \( P_1, P_2, \ldots P_n \) is so.

**Theorem 3.7.** Suppose \( P \) is flat. Then \( P \otimes_R B \equiv B \otimes_R P \) is flat iff \( B \) is flat.

**First proof.** Suppose \( P \) is flat and \( O \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow O \text{ semi-splits}. \) Then, \( P \otimes_R B \) is flat \( \langle = \rangle O \rightarrow P \otimes_R B \otimes_R C' \rightarrow P \otimes_R B \otimes_R C \rightarrow P \otimes_R B \otimes_R C'' \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle O \rightarrow B \otimes_R C' \rightarrow B \otimes_R C \rightarrow B \otimes_R C'' \rightarrow O \text{ semi-splits} \]

\[ \langle = \rangle B \text{ is flat.} \]

**Second proof.** \( O \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow O \text{ is exact, } P \text{ is flat and } P \otimes_R B \) is flat \( \langle = \rangle O \rightarrow P \otimes_R B \otimes_R C' \rightarrow P \otimes_R B \otimes_R C \rightarrow P \otimes_R B \otimes_R C'' \rightarrow O \text{ is exact} \]

\[ \langle = \rangle O \rightarrow B \otimes_R C' \rightarrow B \otimes_R C \rightarrow B \otimes_R C'' \rightarrow O \text{ is exact} \]

\[ \langle = \rangle B \text{ is flat.} \]
Corollary 3.8. \( P_{j_1} \otimes_R P_{j_2} \otimes_R \ldots \otimes_R P_{j_n} \) (where \( j_1, j_2, \ldots, j_n \) is a permutation of \( 1, 2, \ldots, n \)) will be flat whenever each of \( P_1, P_2, \ldots, P_n \) is flat.

Theorem 3.9. Suppose \( F \) is a free module with a finite base. Then \( \text{Hom}_R(F, P) \) is projective iff \( P \) is projective.

First proof.—Suppose \( 0 \to C' \to C \to C'' \to 0 \) semi-splits. Then, \( \text{Hom}_R(F, P) \) is projective \( \iff \) \( 0 \to \text{Hom}_R(\text{Hom}_R(F, P), C') \to \text{Hom}_R(\text{Hom}_R(F, P), C) \to \text{Hom}_R(\text{Hom}_R(F, P), C'') \to 0 \) semi-splits \( \iff \) \( 0 \to F \otimes_R \text{Hom}_R(P, C') \to F \otimes_R \text{Hom}_R(P, C) \to F \otimes_R \text{Hom}_R(P, C'') \to 0 \) semi-splits \( \iff \) \( 0 \to \text{Hom}_R(P, C') \to \text{Hom}_R(P, C) \to \text{Hom}_R(P, C'') \to 0 \) semi-splits \( \iff \) \( P \) is projective.

Second proof. \( 0 \to C' \to C \to C'' \to 0 \) is exact and \( \text{Hom}_R(F, P) \) is projective \( \iff \) \( 0 \to \text{Hom}_R(\text{Hom}_R(F, P), C') \to \text{Hom}_R(\text{Hom}_R(F, P), C) \to \text{Hom}_R(\text{Hom}_R(F, P), C'') \to 0 \) is exact \( \iff \) \( 0 \to F \otimes_R \text{Hom}_R(P, C') \to F \otimes_R \text{Hom}_R(P, C) \to F \otimes_R \text{Hom}_R(P, C'') \to 0 \) is exact \( \iff \) \( P \) is projective.

Corollary 3.10. If \( F \) is a free module with a finite base, then the ring of all endomorphisms of \( F \) over \( R \), viz., \( \text{Hom}_R(F, F) \) is always projective.

Theorem 3.11. If \( \text{Hom}_R(A, B) \) is flat over semi-split sequences of free modules with finite bases, then \( B \) is injective iff \( A \) is so.

Proof. Suppose \( 0 \to F' \to F \to F'' \to 0 \) is a semi-split sequence of free modules with finite bases. Then \( \text{Hom}_R(A, B) \) is flat \( \iff \) \( 0 \to F' \otimes_R \text{Hom}_R(A, B) \to F \otimes_R \text{Hom}_R(A, B) \to F'' \otimes_R \text{Hom}_R(A, B) \to 0 \) semi-splits \( \iff \) \( 0 \to \text{Hom}_R(\text{Hom}_R(F', A), B) \to \text{Hom}_R(\text{Hom}_R(F, A), B) \to \text{Hom}_R(\text{Hom}_R(F'', A), B) \to 0 \) semi-splits.

\[ \therefore \text{B is injective iff A is injective.} \]

Corollary 3.12. If either \( A \) or \( B \) is injective and \( \text{Hom}_R(A, B) \) is flat over semi-split sequences of free modules with finite bases then \( \text{Hom}_R(A, B) \) is injective.

Theorem 3.13. If \( \text{Hom}_R(A, B) \) is flat over exact sequences of free modules with finite bases, then \( B \) is injective iff \( A \) is so.

Proof. Suppose \( 0 \to F' \to F \to F'' \to 0 \) is an exact sequence of free modules with finite bases. Then \( \text{Hom}_R(A, B) \) is flat \( \iff \) \( 0 \to \)
$F' \otimes_R \text{Hom}_R (A, B) \rightarrow F \otimes_R \text{Hom}_R (A, B) \rightarrow F'' \otimes_R \text{Hom}_R (A, B) \rightarrow 0$ is exact $\iff$ $0 \rightarrow \text{Hom}_R (\text{Hom}_R (F', A), B) \rightarrow \text{Hom}_R (\text{Hom}_R (F, A), B) \rightarrow \text{Hom}_R (\text{Hom}_R (F'', A), B) \rightarrow 0$ is exact.

∴ B is injective iff A is injective.

**Corollary 3.14.** If either A or B is injective and $\text{Hom}_R (A, B)$ is flat over exact sequences of free modules with finite bases, then $\text{Hom}_R (A, B)$ is injective.

We now prove a well-known result in a slightly different manner using semi-split sequences.

Suppose F is a free module with a finite base, M an injective module and $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ a semi-split sequence. Then, $0 \rightarrow \text{Hom}_R (C'', M) \rightarrow \text{Hom}_R (C, M) \rightarrow \text{Hom}_R (C', M) \rightarrow 0$ semi-splits $\iff$ $0 \rightarrow F \otimes_R \text{Hom}_R (C'', M) \rightarrow F \otimes_R \text{Hom}_R (C, M) \rightarrow F \otimes_R \text{Hom}_R (C', M) \rightarrow 0$ semi-splits. This, in turn implies, $0 \rightarrow \text{Hom}_R (\text{Hom}_R (F, C''), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C'), M) \rightarrow 0$ semi-splits $\iff$ $0 \rightarrow \text{Hom}_R (F, C') \rightarrow \text{Hom}_R (F, C) \rightarrow \text{Hom}_R (F, C'') \rightarrow 0$ semi-splits $\iff$ F is projective.

This proves,

**Theorem 3.15.** A free module with a finite base is projective.

**Remark 3.16.** Theorem 3.15 can also be proved by considering exact sequences in a similar manner and for the sake of completeness the proof is given below:

If F is free with a finite base, M is injective and $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ an exact sequence, then, $0 \rightarrow \text{Hom}_R (C'', M) \rightarrow \text{Hom}_R (C', M) \rightarrow \text{Hom}_R (C, M) \rightarrow \text{Hom}_R (C', M) \rightarrow \text{Hom}_R (C, M) \rightarrow 0$ is exact $\iff$ $0 \rightarrow F \otimes_R \text{Hom}_R (C'', M) \rightarrow F \otimes_R \text{Hom}_R (C', M) \rightarrow F \otimes_R \text{Hom}_R (C, M) \rightarrow F \otimes_R \text{Hom}_R (C', M) \rightarrow F \otimes_R \text{Hom}_R (C, M) \rightarrow 0$ is exact $\iff$ $0 \rightarrow \text{Hom}_R (\text{Hom}_R (F, C''), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C'), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C'), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C), M) \rightarrow \text{Hom}_R (\text{Hom}_R (F, C''), M) \rightarrow 0$ is exact $\iff$ F is projective.

Thus, we observe from the proofs of the above theorems that semi-split sequences and exact sequences yield the same results on certain subcategories of the category of all modules.

4. In this section we consider a special case and assume that R is Noetherian.
Let $A$ be a finitely generated module over $R$ and $B$ any injective module. Then there exists the module isomorphism.

$$\text{Tor}_n^R(A, \text{Hom}_R(C, B)) \cong \text{Hom}_R(\text{Ext}_R^n(A, C), B)$$

for all modules $C$.

Suppose now that $A$ is flat. Then,

$$\text{Tor}_n^R(A, D) = 0, \forall n \geq 1,$$

for any module $D$.

\[ \therefore \text{in particular, } \forall n \geq 1, \]

$$\text{Tor}_n^R(A, \text{Hom}_R(C, B)) = 0,$$

for all modules $C$.

\[ \therefore \text{Ext}_n^R(A, C) = 0, \forall n \geq 1, \text{ for all modules } C. \]

This completes the proof of

Theorem 4.1. If $A$ is a finitely generated flat module, then

$$\text{Ext}_n^R(A, C) = 0, \forall n \geq 1,$$

for all modules $C$.

Corollary 4.2. $w.dh_R(A) = dh_R(A) \leq 0$, for any finitely generated flat module $A$.

5. We end this paper with a few remarks which are, of course, trivial, but are included in this since the author has not seen them elsewhere.

Remark 5.1. As a consequence of the result “A projective module is flat,” $P \otimes_R -$ will be an exact covariant functor whenever $\text{Hom}_R(P, -)$ is so. From this we deduce the result ‘$P \otimes_R -$ and $- \otimes_R P$ will be exact functors when $P$ is projective’ (See [4], Theorem 4, page 66).

Remark 5.2. A module $P$ is projective iff every exact sequence of the form $A \rightarrow B \rightarrow P \rightarrow 0$ splits (See [1], proposition 2.4, page 7). The condition that $A \rightarrow B$ must be a monomorphism is not necessary.

Remark 5.3. A module $Q$ is injective iff every exact sequence of the form $0 \rightarrow Q \rightarrow A \rightarrow B$ splits. (See [1], proposition 3.4, page 10). The condition that $A \rightarrow B$ must be an epimorphism is not necessary.
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