ON CERTAIN DOUBLE GENERALISED TRANSFORM

By V. M. Pandey and R. N. Pandey
(*Institute of Technology, Banaras Hindu University*)

Received September 25, 1970
(Communicated by Prof. R. S. Mishra, F.A.Sc.)

1. Introduction

Rainville[4] (p. 104) has shown that the single and double Euler transformations of the hypergeometric function ${}_pF_q$ are effective tools for augmenting its parameters. The double Whittaker transform of generalised hypergeometric function given by Srivastava and Joshi[6] has similar interesting properties.

Recently, Srivastava and Singhal[5] have discussed a double Meijer transform of the generalised hypergeometric function in the form

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^{\mu+\frac{1}{2}} e^{-\frac{1}{2}\lambda(x+y)} K_\nu [\tfrac{1}{2}\lambda(x+y)]$$

$$\times {}_pF_q \left[\begin{matrix} a_1 \cdots a_p; \\ b_1 \cdots b_q; \end{matrix} \; tx^s y^k (x+y)^r \right] dxdy$$

$$= \frac{\sqrt{\pi}\, \Gamma(\alpha+\beta+\mu\pm\nu+\frac{1}{2})}{\lambda^{\alpha+\beta+\mu+\frac{1}{2}}\, \Gamma(\alpha+\beta+\mu+1)} B(\alpha, \beta)$$

$$\times {}_{p+3s+3k+2r}F_{q+2s+2k+r} \left[\begin{matrix} a_1 \cdots a_p, \triangle(s, \alpha), \triangle(k, \beta), \\ b_1 \cdots b_q, \triangle(s+k, \alpha+\beta), \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle(s+k+r, \alpha+\beta+\mu\pm\nu+\frac{1}{2}; \\ \triangle(s+k, \alpha+\beta+\mu+1); \end{matrix} \; t\delta\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}\right], \tag{1.1}$$

where r, s, k are non-negative integers; $R(\alpha) > 0$, $R(\beta) > 0$, $R(\lambda) > 0$ and $R(\alpha+\beta+\mu\pm\nu+\frac{1}{2}) > 0$; $\triangle(s, \alpha)$ stands for a set of 's' parameters.

$$\frac{a}{s}, \frac{a+1}{s}, \frac{a+2}{s}, \ldots \frac{a+s-1}{s}; \mathrm{B}(a, \beta) \text{ stands for } \frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)}$$

and for sake of brevity, the gamma-product $\Gamma(a-\beta)\Gamma(a+\beta)$

is denoted by

$$\Gamma(a \pm \beta)$$

and

$$\delta = \frac{s^s k^k}{(s+k)^{s+k}}.$$

Here we present a double generalised transform of the Meijer's G-function which leads to yet another interesting process of augmenting the parameters in the Meijer's G-function. The usefulness of the generalised operator which we introduce in the next section provides an alternative derivation of Weisner's bilateral generating function and its numerous other applications to certain classical polynomials.

2. THE GENERAL FORMULA

We start with the following result[1] (p. 177):

$$\int_0^\infty \int_0^\infty \psi(x+y)\, x^{a-1} y^{\beta-1}\, dxdy = \mathrm{B}(a, \beta) \int_0^\infty \Psi(z)\, z^{a+\beta-1}\, dz \qquad (2.1)$$

which holds when $\mathrm{R}(a) > 0$, $\mathrm{R}(\beta) > 0$.

We consider

$$\int_0^\infty \int_0^\infty \phi(x+y)\, x^{a-1} y^{\beta-1}\, G_{e_1, e_2}^{e_3, e_4}\left(-tx^s y^k (x+y)^r \,\middle|\, \begin{matrix} a_1 \cdots a_{e_1} \\ b_1 \cdots b_{e_2} \end{matrix}\right) dxdy, \qquad (2.2)$$

where $\mathrm{R}(a) > 0$, $\mathrm{R}(\beta) > 0$, s, k, r, are positive integers. To evaluate (2.2), we express the G-function as a Mellin-Barnes type integral [(2), p. 207 (1)] and then using (2·1), we obtain

$$\int_0^\infty \int_0^\infty \phi(x+y)\, x^{a-1} y^{\beta-1}\, G_{e_1, e_2}^{e_3, e_4}\left(-tx^s y^k (x+y)^r \,\middle|\, \begin{matrix} a_1, \ldots, a_{e_1} \\ b_1, \ldots, b_{e_2} \end{matrix}\right) dxdy$$

$$= \frac{\sqrt{2\pi} s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{(s+k)^{\alpha+\beta-\frac{1}{2}}} \int_0^\infty \phi(z)\, z^{\alpha+\beta-1}\, G^{e_3,\, e_4,+s+k}_{e_1+s+k,\, e_2+s+k}$$

$$\times \left(\frac{-tz^{s+k+r} s^s k^k}{(s+k)^{s+k}} \middle| \begin{matrix} a_1 \cdots a_{e_1},\ \triangle(s, 1-\alpha),\ \triangle(k, 1-\beta) \\ b_1, \cdots b_2,\ \triangle[s+k, 1-(\alpha+\beta)] \end{matrix} \right)$$

$$\times dz \tag{2.3}$$

provided that $R(\alpha) > 0$, $R(\beta) > 0$ and the integral on the right-hand side of (2.3) is absolutely convergent.

3. In this section we define our generalised double transform of $\psi(x, y)$ as

$$O^{e_1', e_2', e_3', e_4'}_{\alpha, \beta, \sigma, \lambda} \{\psi(x, y)\}$$

$$= A \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} G^{e_3', e_4'}_{e_1', e_2'} \left(\lambda(x+y) \middle| \begin{matrix} c_1 \cdots c_{e_1'} \\ d_1 \cdots d_{e_2'} \end{matrix} \right)$$

$$\times \psi(x, y)(x+y)^\sigma\, dxdy, \tag{3.1}$$

provided that $R(\alpha) > 0$, $R(\beta) > 0$ and the double integral is absolutely convergent, and $A = f(\alpha, \beta, \lambda, \sigma; c_1, \cdots, c_{e_1'}; d_1, \cdots d_{e_2'})$. The double generalised transform (3.1) of the Meijer's G-function is obtained by putting

$$\psi(x, y) = G^{e_3\, e_4}_{e_1\, e_2} \left(-tx^s y^k (x+y)^r \middle| \begin{matrix} a_1, \cdots, a_{e_1} \\ b_1, \cdots, b_{e_2} \end{matrix} \right).$$

Thus, we get, on using (2.3),

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^\sigma G^{e_3', e_4'}_{e_1', e_2'} \left(\lambda(x+y) \middle| \begin{matrix} c_1, \cdots, c_{e_1'} \\ d_1, \cdots, d_{e_2'} \end{matrix} \right)$$

$$\times G^{e_3, e_4}_{e_1, e_2} \left(-tx^s y^k (x+y)^r \middle| \begin{matrix} a_1, \cdots, a_{e_1} \\ b_1, \cdots, b_{e_2} \end{matrix} \right) dxdy$$

$$= \frac{\sqrt{2\pi}\, s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}} \lambda^{-(\alpha+\beta+\sigma)}}{A(s+k)^{\alpha+\beta-\frac{1}{2}}} H^{e_3+e_4', e_4+s+k+e_3'}_{e_1+s+k+e_2', e_2+s+k+e_1'}$$

$$\times \left[\frac{-t\delta}{\lambda^{r+k+s}} \middle| \begin{matrix} \{(a_{e_4}, 1)\},\ \{1-d_{e_2'}-(\alpha+\beta+\sigma), r+k+s\}, \\ \{(b_{e_3}, 1)\},\ \{1-c_{e_1'}-(\alpha+\beta+\sigma), r+k+s\}, \end{matrix} \right.$$

$$[\triangle(s, 1-\alpha), 1], [\triangle(k, 1-\beta), 1], \{\triangle(s+k, 1-\overline{\alpha+\beta}), 1)\},$$

$$\left.\begin{matrix}(a_{e_4+1}, 1), \cdots (a_{e_1}, 1) \\ (b_{e_3+1}, 1), \cdots (b_{e_2}, 1)\end{matrix}\right], \tag{3.2}$$

provided that

$$R[\alpha+\beta+\sigma+(r+k+s)\,b_h+d_i]>0\;(h=1,\cdots e_3;$$

$$i=1,\cdots, e_3),\; R[\alpha+\beta+\sigma+(c_j-1)+(r+k+s)(a_{h'}-1)]<0$$

$$(j=1,\ldots,e_4;\; h'=1,\ldots,e_4+s+k),$$

$$a_{h'}=\{(a_{e_4})\},\; \triangle(s, 1-\alpha),\; \triangle(k, 1-\beta),$$

$$2(e_3+e_4)-e_1-e_2\equiv\lambda_1>0,\; 2(e_3'+e_4')-e_1'-e_2'\equiv\mu>0,$$

$$|\arg t\delta|<\tfrac{1}{2}\lambda_1\pi \quad\text{and}\quad |\arg\lambda|<\tfrac{1}{2}\mu\pi.$$

Particular case of (3.2)

On putting $e_3=1$, $e_1=e_4=p$, $e_2=q+1$, $a_{e_1}=1-A_{e_1}\,(e_1=1,\ldots p)$, $b_1=0$, $b_{j+1}=1-B_j\,(j=1,\ldots q)$, $e_3'=2=e_2'$, $e_4'=0$, $e_1'=1=c$, $d_1=\nu+\frac{1}{2}$ and $d_2=-\nu+\frac{1}{2}$ in (3.2), we obtained (1.1).

4. APPLICATION

(*a*) *Scme generating relation.*—In the Bateman generating function[4] (p. 256)

$${}_0F_1[-;\,1+\alpha;\,\tfrac{1}{2}(x-1)\,t]\,{}_0F_1[-;\,1+\beta;\,\tfrac{1}{2}(x+1)\,t]$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{(1+\alpha)_n(1+\beta)_n}P_n^{(\alpha,\beta)}(x),$$

if we replace t by tuv, taking u and v as variables and operating both sides by

$$O_{\alpha',\beta',\sigma,\lambda}^{e_1',e_2',e_3',e_4'},$$

we get

$$2^{\left\{(e_2'-e_1')(\alpha'+\beta'+\sigma)+e_1'-e_4'-e_3'+\sum_{j=1}^{e_3'} {}_j-\sum_{j=1}^{e_1'} c_j-\alpha'-\beta'+1\right\}}$$

$$\times\ \pi^{-e_3'-e_4'+\frac{1}{2}e_1'+\frac{1}{2}e_2'} \sum_{n=0}^{\infty} \frac{[\frac{1}{2}t(x-1)]^n\,\Gamma(\alpha'+n)\,\Gamma(\beta'+n)\,\lambda^{-2n}}{n!\,(1+\alpha)_n\,\Gamma\left(\frac{\alpha'+\beta'}{2}+n\right)\Gamma\left(\frac{\alpha+\beta'+1}{2}+n\right)}$$

$$\times\ \frac{\prod_{j=1}^{e_3'} \Gamma\{\frac{1}{2}(d_j+\alpha'+\beta'+\sigma+2n)\}\,\Gamma\{\frac{1}{2}(d_j+\alpha'+\beta'+\sigma+1+2n)\}}{\prod_{j=e_3'+1}^{e_2'} \Gamma\{\frac{1}{2}(1-d_j-\alpha'-\beta'-\sigma-2n)\}\,\Gamma\{\frac{1}{2}(2-d_j-\alpha'-\beta'-\sigma-2n)\}}$$

$$\times\ \frac{\prod_{j=1}^{e_4'} \Gamma\{\frac{1}{2}(1-c_j-\alpha'-\beta'-\sigma-2n)\}\,\Gamma\{\frac{1}{2}(2-c_j-\alpha'-\beta'-\sigma-2n)\}}{\prod_{j=e_4'+1}^{e_1'} \Gamma\{\frac{1}{2}(c_j+\alpha'+\beta'+\sigma+2n)\}\,\Gamma\{\frac{1}{2}(c_j+\alpha'+\beta'+\sigma+1+2n)\}}$$

$$\times\ {}_{2e_2'+2}F_{2e_1'+3}\left[\begin{matrix}\alpha'+n,\ \beta'+n,\ \{\frac{1}{2}(d_{e_2'}+\alpha'+\beta'+\sigma+2n)\}, \\ 1+\beta,\ \left(\frac{\alpha'+\beta'}{2}+n\right),\ \left(\frac{\alpha'+\beta'+1}{2}+n\right),\ \left\{\frac{1}{2}c_{e_1'}+\alpha'+\beta'+\sigma+2n\right\},\end{matrix}\right.$$

$$\left.\begin{matrix}\left\{\frac{1}{2}(d_{e_2'}+\alpha'+\beta'+\sigma+1+2n)\right\}; \\ \left\{\frac{1}{2}(c_{e_1'}+\alpha'+\beta'+\sigma+1+2n)\right\};\end{matrix}\ \frac{2^{2(e_2'-e_1'-1)}}{\lambda^2}\left\{\frac{1}{2}t(x+1)\right\}\right]$$

$$=\sum_{n=0}^{\infty} \frac{t^n\,P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n(1+\beta)_n}\,B(\alpha'+n,\ \beta'+n)\,\lambda^{-2n}$$

$$\times\ \frac{\prod_{j=1}^{e_3'} \Gamma(d_j+\alpha'+\beta'+2n+\sigma)\ \prod_{j=1}^{e_4'} \Gamma(1-c_j-\alpha'-\beta'-2n-\sigma)}{\prod_{j=e_3'+1}^{e_2'} \Gamma(1-d_j-\alpha'-\beta'-2n-\sigma)\ \prod_{j=e_4'+1}^{e_1'} \Gamma(c_j+\alpha'+\beta'+2n+\sigma)}, \qquad (4.1)$$

provided that

$$e_1'+e_2'<2(e_3'+e_4'),\ |\arg\lambda|<(e_3'+e_4'-\tfrac{1}{2}e_1'-\tfrac{1}{2}e_2')\pi$$

and

$$-\min_{1\le j\le e_3'} R(d_j)<R(\alpha'+\beta'+\sigma)<1-\max_{1\le j\le e_4'} R(c_j),$$

Particular case

If we put $e_1' = 1$, $e_2' = 2 = e_3'$, $e_4' = 0 = c_1 = d_2$, $\lambda = 1 = d_1$ and $\sigma = -1$ in (4.1), we get

$$\sum_{n=0}^{\infty} \frac{(\alpha')_n (\beta')_n}{(1+\alpha)_n (1+\beta)_n} t^n \mathrm{P}_n^{(\alpha,\beta)}(x)$$

$$= \mathrm{F}_4 [\alpha', \beta'; 1+\alpha, 1+\beta; \tfrac{1}{2} t(x-1), \tfrac{1}{2} t(x+1)]$$

a formula due to Brafman[4] (p. 271), where F_4 is Appell's function.

If, however, we replace t by tu and regard u as variable the Bateman generating function, when operated by

$$\mathrm{O}_{\alpha', \beta', \sigma, \lambda}^{e_1', e_2', e_3' e_4'}$$

leads to a result

$$\sum_{n=0}^{\infty} \frac{\lambda^{-n} [\frac{1}{2} t(x-1)]^n (\alpha')_n}{n! (1+\alpha)_n (\alpha'+\beta')_n}$$

$$\times \frac{\prod_{j=1}^{e_3'} \Gamma(d_j + \alpha' + \beta' + n + \sigma) \prod_{j=1}^{e_4'} \Gamma(1 - c_j - \alpha' - \beta' - \sigma - n)}{\prod_{j=e_3'+1}^{e_2'} \Gamma(1 - d_j - \alpha' - \beta' - n - \sigma) \prod_{j=e_4'+1}^{e_1'} \Gamma(c_j + \alpha' + \beta' + n + \sigma)}$$

$$\times {}_{e_2'+1}\mathrm{F}_{e_1'+2} \left[\begin{matrix} n+\alpha', \{(d_{e_2'} + \alpha' + \beta' + n + \sigma)\}; \\ 1+\beta, (\alpha'+\beta'+n), \{(c_{e_1'} + \alpha' + \beta' + n + \sigma)\}; \end{matrix} \ \frac{1}{2} \frac{t}{\lambda}(x+1) \right]$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^{-n} \mathrm{P}_n^{(\alpha,\beta)}(x) t^n (\alpha')_n}{(1+\alpha)_n (1+\beta)_n}$$

$$\times \frac{\prod_{j=1}^{e_3'} \Gamma(d_j + \alpha' + \beta' + n + \sigma) \prod_{j=1}^{e_4'} \Gamma(1 - c_j - \alpha' - \beta' - n - \sigma)}{\prod_{j=e_3'+1}^{e_2'} \Gamma(1 - d_j - \alpha' - \beta' - n - \sigma) \prod_{j=e_4'+1}^{e_1'} \Gamma(c_j + \alpha' + \beta' + \sigma + n)}, \qquad (4.2)$$

provided that conditions given in (4.1) are satisfied.

Particular case

If we have $e_2' = 2 = e_3'$, $e_1' = \lambda = 1 = d_1$, $e_4' = d_2 = 0$, $c_1 = 0$ and $\sigma = -1$ in (4.2), we get

$$\psi_2[\alpha', 1+\alpha, 1+\beta; \tfrac{1}{2}(x-1)\,t, \tfrac{1}{2}(x+1)\,t]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha')_n\, t^n\, P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n\,(1+\beta)_n},$$

where ψ_2 is a confluent hypergeometric function in two arguments defined by Humbert[2] (p. 225).

(*b*) *Bilateral generating function.*—Let us consider the Hille-Hardy formula[4] (p. 212)

$$(1-t)^{-1-\alpha}\exp.\left[\frac{-t(x+y)}{1-t}\right]{}_0F_1\left[-\,;\,1+\alpha\,;\,\frac{xyt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(y)\, t^n,$$

which can be written as

$$(1-t)^{-1-\alpha}\exp.\left[\frac{-tx}{1-t}\right] \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-)^r \frac{(ty)^{r+s}\, x^s}{r!\; s!\; (1-t)^{r+s}(1+\alpha)_s}$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(y)\, t^n\, L_n^{(\alpha)}(x).$$

If we replace y to uy and operating both sides by

$$O^{e_1', e_2', e_3', e_4'}_{\alpha', \beta', \sigma, 1},$$

we arrive at

$$(1-t)^{-1-\alpha}\exp.\left[\frac{-tx}{1-t}\right] \sum_{s=0}^{\infty} \frac{(tux)^s}{s!\,(1-t)^s\,(1+\alpha)_s}\,\frac{(\beta')_s\,\lambda^{-s}}{(\alpha'+\beta')_s}$$

$$\times \frac{\prod_{j=1}^{e_3'} \Gamma(d_j+\alpha'+\beta'+s+\sigma) \prod_{j=1}^{e_4'} \Gamma(1-c_j-\alpha'-\beta'-\sigma-s)}{\prod_{j=e_3'+1}^{e_3'} \Gamma(1-d_j-\alpha'-\beta'-\sigma-s) \prod_{j=e_4'+1}^{e_1'} \Gamma(c_j+\alpha'+\beta'+\sigma+s)}$$

$$\times\ {}_{e_2'+1}F_{e_1'+2}\left[\begin{matrix}\beta'+s,\ \{(d_{e_2'}+\alpha'+\beta'+\sigma+s)\}; \\ \alpha'+\beta'+s,\ \{(c_{e_1'}+\alpha'+\beta'+\sigma+s)\};\end{matrix}\quad -\frac{ut}{1-t}\right]$$

$$=\sum_{n=}^{\infty} L_n^{(\alpha)}(x)\,\frac{\prod_{j=1}^{e_3'}\Gamma(d_j+\alpha'+\beta'+\sigma)\prod_{j=1}^{e_4'}\Gamma(1-c_j-\alpha'-\beta'-\sigma)}{\prod_{j=e_3'+1}^{e'_2}\Gamma(1-d_j-\alpha'-\beta'-\sigma)\prod_{j=e_4'+1}^{e_1'}\Gamma(c_j+\alpha'+\beta'+\sigma)}$$

$$\times\ {}_{e_2'+2}F_{e_1'+2}\left[\begin{matrix}-n,\beta,\ \{(d_{e_2'}+\alpha'+\beta'+\sigma)\}; \\ 1+\alpha,\ \alpha'+\beta',\ \{(c_{e_1'}+\alpha'+\beta'+\sigma)\};\end{matrix}\quad u\right]$$

valid for conditions given in (4.1).

Particular case

If we put $e_1'=1=d_1$, $e_2'=2=e_3'$, $e_4'=0=c_1=d_2$, $\sigma=-1$, $\alpha'=c$ and $\beta'=\beta$ in (4.3), we get

$$(1-t)^{-1-\alpha}\exp.\left[-\frac{tx}{(1-t)}\right]\sum_{s=0}^{\infty}\frac{(\beta)_s\,(tux)^s}{s!\,(1+\alpha)_s(1-t)^s}\left[1+\frac{ut}{1-t}\right]^{-\beta-s}$$

$$=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,{}_2F_1\left[\begin{matrix}-n,\beta; \\ 1+\alpha;\end{matrix}\ u\right]t^n.$$

ι finally simplifies to the well-known formula due to Weisner[7]

$$(1-t)^{\alpha+\beta-1}\exp.\left[-\frac{tx}{1-t}\right](1-t+ut)^{-\beta}\,{}_1F_1\left[\beta;\ 1+\alpha;\ \frac{xut}{(1-t)(1-t+ut)}\right]$$

$$=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,{}_2F_1\left[\begin{matrix}-n,\beta; \\ 1+\alpha;\end{matrix}\ u\right]t^n$$

REFERENCES

1. Edward, J. .. *A Treatise on the Integral Calculus*, Chelsea, New York, 1954.

2. Erdelyi, A. .. *Higher Transcedental Functions*, Vol. 1, Bateman Manuscript Project, McGraw-Hill Co., 1953.

3. ——— .. *Ibid.*, Vol. 2, Bateman Manuscript Project, McGraw-Hill Co., 1956.

4. Rainville, E. D. .. *Special Functions*, MacMillan, New York, 1960.

5. Srivastava, H. M. and Singhal, J. P. "Double Meijer transformations of certain hypergeometric functions," *Proc. Cambridge Philos. Soc.*, 1968, **64** (2), 425–30.

6. Srivastava, H. M. and Joshi, C. M. "Certain double Whittaker transforms of generalised hypergeometric functions," *The Yokohama Mathematical Journal*, Oct. 1967, **15** (1).

7. Weisner, L. .. "Group theoretic origin of certain generating functions," *Pacific J. Math.*, 1955, pp. 1033–39.