SOME NEW RESULTS ON ε-TENSOR PRODUCTS
OF LOCALLY CONVEX SPACES

BY KAMAL KANT JHA

(Lecturer in Mathematics, Patna University, Science College, Patna-5, India)

Received July 28, 1969

(Communicated by Dr. N. S. Nagendra Nath, F.A.Sc.)

ABSTRACT

The paper contains the proofs of three new propositions on ε-tensor products of locally convex spaces. The first two of these propositions are on the ε-tensor product of inductive limits of locally convex spaces. The third proposition is on integral bilinear forms. For inductive tensor products and π-tensor products, some results on properties of permanence appear in A. Grothendieck’s famous thesis. We prove, in the present paper, some properties of permanence of ε-tensor products.

NOTATIONS AND FUNDAMENTAL IDEAS

1. Let $E \times F$ denote the algebraic tensor product of vector spaces $E$ and $F$ over the same field $K$. Then every element of $E \times F$ can be written as a sum

\[ \sum_{i=1}^{n} x_i \times y_i, \; x_i \in E, \; y_i \in F. \]

2. If $A$ and $B$ are subsets of vector spaces $E$ and $F$ respectively, we define the set

\[ A \times B = \{ a \times b : a \in A, \; b \in B \}. \]

Let us note that the tensor product $E \times F$ of vector spaces $E$ and $F$ is not exhausted by elements of the form $x \times y$, it also contains elements which are finite sums

\[ \sum_{i=1}^{n} x_i \times y_i, \; x_i \in E, \; y_i \in F. \]

3. Tensor product of linear maps.—Let $E_1$, $F_1$, $E$, $F$ be four vector spaces over the same field $K$. Let $u : E_1 \rightarrow E$ and $v : F_1 \rightarrow F$ be linear
mappings. Then there is a unique linear map of $E_1 \times F_1$ into $E \times F$, called the tensor product of $u$ and $v$ and denoted by $u \otimes v$, such that

$$u \otimes v(x \otimes y) = u(x) \otimes v(y)$$

for all $x \in E_1$, $y \in F_1$.

4. If $(E, F)$ is a dual pair of vector spaces and if $A$ is a subset of $E$, then $A^0$ denotes the polar of $A$ in $F$. Also $\Gamma A$ denotes the absolutely convex envelope of $A$, i.e., $\Gamma A$ is the set of all finite linear combinations $\sum a_i x_i$ with $\sum |a_i| \leq 1$ and $x_i \in A$.

5. Theorem of bipolars.---If the vector spaces $E$ and $F$ over the same field $K$ form a pairing with respect to a bilinear form

$$(x, y) \mapsto B(x, y),$$

and if $A$ is a non-empty subset of $E$, then $A^{00}$, that is the bipolar of $A$ is the absolutely convex weakly closed envelope of $A$, i.e., the smallest absolutely convex set containing $A$ which is closed for the weak topology $s(E, F)$. We write $A^{00} = \overline{\Gamma A}$.

6. A corollary of Hahn-Banach theorem.---Let $E$ and $F$ be two locally convex topological vector spaces.

Let $u$ be a continuous linear map of $E$ into $F$ and $u'$ denote the transpose of $u$. Then the following two conditions are equivalent:

(a) $u(E)$ is dense in $F$;
(b) $u'$ is one-to-one.

7. Theorem of Dieudonné-Schwartz.---Let $E$ and $F$ be two Frechet spaces with topologies $T_E$ and $T_F$ respectively, $E'$ and $F'$ their duals, and let $u: E \rightarrow F$ be a linear map. Then the following conditions are equivalent:

(a) $u$ is a strict morphism for $T_E$ and $T_F$;
(b) $u$ is a strict morphism for $s(E, E')$ and $s(F, F')$.

(γ) $u(E)$ is closed in $F$;
(δ) $u'(F')$ is closed in $E'$ for $s(E', F)$ and $s(E', E)$;
(ε) $u'(F')$ is closed in $E'$ for $s(E', E)$.

8. Let $E$ be the inductive limit of the locally convex topological vector spaces $(E_i)_{i \in I}$ by the linear mappings $(u_i)_{i \in I}$. If for each $i \in I$, $D_i$ is a base
Some New Results on $\epsilon$-Tensor Products of Locally Convex Spaces

of absolutely convex neighbourhoods of origin in $E_i$, then the set $D$ of absolutely convex envelopes of sets of the form

$$\bigcup_{i \in I} \bigcup_{i \in D_i} V_i$$

form a base of absolutely convex neighbourhoods of origin in $E$ for the inductive limit topology.

9. Let $(E_i)_{i \in I}$ be a family of locally convex Hausdorff spaces. Then the dual of the product space $\pi E_i$ can be canonically identified with the external direct sum $\sum_{i \in I} E_i'$ of the dual spaces $E_i'$.

10. Let $E$ and $F$ be locally convex spaces. Let $L(E, F)$, $E^*$, $L(E, F)$, $E'$ denote respectively the space of linear maps $E \to F$, space of all linear forms on $E$, space of all continuous linear maps $E \to F$, and the space of all continuous linear forms on $E$. Let $B(E \times F)$ be the space of all bilinear forms on $E \times F$. Then the following canonical isomorphisms hold:

$$B(E \times F) \cong L(E, F^*) \cong L(F, E^*)$$

$$B \leftrightarrow \hat{B} \leftrightarrow \tilde{B}.$$  

Let $b(E \times F)$ be the space of all separately continuous bilinear forms on $E \times F$ and $B(E \times F)$ be the space of all continuous bilinear forms on $E \times F$. Then

$$b(E \times F) \cong L(E, F_s') \cong L(F, E_s')$$

$$B \leftrightarrow \hat{B} \leftrightarrow \tilde{B}$$

where $E_s'$ and $F_s'$ denote $E'$ and $F'$ with simple topology. Further we have

$$B(E \times F) \cong (E \times F)^* \cong L(E, F^*) \cong L(F, E^*)$$

$$B(x, y) = \hat{B}(x \times y) = \tilde{B}(x)(y) = \tilde{B}(y)(x).$$

Introduction.—Let $E$ and $F$ be locally convex (topological vector) spaces. A locally convex topology $T$ on $E \times F$ is said to be compatible with the tensor product if and only if

(a) the canonical map $\psi: E \times F \to E \times F_T$ is separately continuous, where $E \times F$ has the product topology and $E \times F_T$ denotes the tensor product $E \times F$ with the topology $T$;
(b) \( \mu, v \in E', \nu \in F' \) implies \( \mu \times v \) is continuous linear form on \( E \times_T F \);

(c) if \( G \) and \( H \) are any equicontinuous subsets of \( E' \) and \( F' \) respectively, then \( G \times H \) is an equicontinuous subset of \( (E \times_T F)' \).

Observation I.—Condition (a) is equivalent to the requirement

\[ (E \times_T F)' \subseteq b(E \times F). \]

Proof.—Let us assume first condition (a).

Let \( x_0 \) be a fixed point of \( E \) and let \( W \) be an arbitrary neighbourhood of origin in \( E \times_T F \). Then there exists a neighbourhood \( V \) of origin in \( F \) with \( \psi(x_0, V) = x_0 \times V \subseteq W \).

Hence if

\[ \hat{B} \in (E \times_T F)' \text{ with } |\hat{B}(W)| \leq 1, \text{ then } |B(x_0, V)| \leq 1 \text{ and } B \text{ is separately continuous.} \]

Next let us assume that \( (E \times_T F)' \subseteq b(E \times F) \). Let us select \( \hat{B} \in (E \times_T F)' \) and let \( W \) be such that \( |\hat{B}(W)| \leq 1 \). Given \( x_0 \in E \), the assumption implies that there exists a neighbourhood \( V \) of origin in \( F \) such that \( |B(x_0, V)| \leq 1 \) and thus also

\[ |\hat{B}(x_0 \times V)| \leq 1. \]

Hence \( x_0 \times V \subseteq W \). Hence \( \psi(x_0, V) = x_0 \times V \subseteq W \) and \( \psi \) is separately continuous.

Observation II.—Condition (b) is equivalent to the requirement

\[ E' \times F' \subseteq (E \times_T F)'. \]

Hence it follows from observations I and II that the conditions (a) and (b) of the definition of a compatible topology on \( E \times F \), are equivalent to the requirement

\[ E' \times F' \subseteq (E \times_T F)'. \]

Theorem.—A necessary and sufficient condition for a locally convex topology \( T \) on \( E \times F \) to be compatible with the tensor product is that \( T \) is a topology of uniform convergence on a class \( m \) of subsets \( M \) of a space \( H \subseteq b(E \times F) \) satisfying the following two conditions.
(i) For every \( x_0 \in \mathcal{E} \), \( \tilde{M}(x_0) \) is equicontinuous in \( \mathcal{F}' \);

For every \( y_0 \in \mathcal{F} \), \( \tilde{M}(y_0) \) is equicontinuous in \( \mathcal{E}' \).

(ii) \( m \) contains the class \( m_0 \) of all sets \( G \times H \), where \( G \) and \( H \) are equicontinuous subsets of \( \mathcal{E}' \) and \( \mathcal{F}' \) respectively.

Proof.—Necessity. Let us assume that \( \mathcal{T} \) is compatible. Using observation I we have

\[
(\mathcal{E} \times \mathcal{F})' \subset b(\mathcal{E} \times \mathcal{F}).
\]

\( \mathcal{T} \) is the topology of uniform convergence on equicontinuous subsets of \( (\mathcal{E} \times \mathcal{F})' \subset b(\mathcal{E} \times \mathcal{F}) \).

Condition (ii) is clearly satisfied since the sets \( G \times H \) are equicontinuous by condition (c) of the definition.

In order to verify condition (i), we use condition (a) of the definition. Let \( x_0 \in \mathcal{E} \) be given. Now \( M \) equicontinuous on \( \mathcal{E} \times \mathcal{F} \) implies that there exists a neighbourhood \( W \) in \( \mathcal{E} \times \mathcal{F} \) with \( |\tilde{M}(W)| \leq 1 \). By condition (a) of definition, we select a neighbourhood \( V \) in \( \mathcal{F} \) and a real number \( \rho \) with \( \rho x_0 \times V \subseteq W \).

Then \( |\tilde{M}(\rho x_0 \times V)| \leq 1 \) or equivalently

\[
|\tilde{M}(\rho x_0)(V)| \leq 1 \quad \text{or} \quad \tilde{M}(\rho x_0) \subseteq V^0,
\]

and therefore \( \tilde{M}(x_0) \) is equicontinuous.

Conversely, suppose that the conditions given in the theorem hold. It follows immediately that the conditions of the definition are also satisfied.

Remarks.—The weakest compatible topology on \( \mathcal{E} \times \mathcal{F} \) is the topology of uniform convergence on the class of all sets \( G \times H \) in \( \mathcal{E}' \times \mathcal{F}' \) where \( G \) and \( H \) are equicontinuous in \( \mathcal{E}' \) and \( \mathcal{F}' \) respectively. This topology is called the \( \varepsilon \)-topology and \( \mathcal{E} \times \mathcal{F} \) with this topology is denoted by \( \mathcal{E} \times_{\varepsilon} \mathcal{F} \).

We shall denote by \( \mathcal{E} \times_{\varepsilon} \mathcal{F} \), the complete \( \varepsilon \)-tensor product.

It is easy to see that \( \mathcal{E} \times_{\varepsilon} \mathcal{F} \) is identifiable with the space \( \mathcal{B}_{\varepsilon}(\mathcal{E}' \times \mathcal{F}') \) of continuous bilinear forms on \( \mathcal{E}' \times \mathcal{F}' \), equipped with the topology of uniform convergence on the products \( G \times H \); \( G \), \( H \) being equicontinuous.
subsets of $E'$ and $F'$ respectively. It also follows easily that there exists a
continuous injection of the dual of $E \times F$ into $\mathcal{B}(E \times F)$.

**Definition.**—The canonical image of the dual of $E \times F$ into $\mathcal{B}(E \times F)$
is denoted by $J(E \times F)$; its elements are called the integral forms on $E \times F$.

Now we prove the following results on the $\varepsilon$-tensor product of locally
convex spaces.

**Proposition 1.**—Let $E$ be a locally convex space which is the inductive
limit of a family $(E_i)$ of locally convex (not necessarily Hausdorff) spaces
by the linear mappings $u_i$. Let $F$ be a locally convex space which is the
inductive limit of a family $(F_j)$ of locally convex (not necessarily Hausdorff)
spaces by the linear mappings $v_j$. Let $\tau$ denote the finest convex topology
on $E \times F$ for which the linear maps $u_i \times v_j : E_i \times F_j \to E \times F$ are con-
tinuous and let $E \times F$ with such a topology $\tau$ be denoted by $E \times F$.

Then every equicontinuous subset $M$ of $(E \times F)'$ is also an equiconti-
nuous subset of $(E \times F)'$.

**Proof.**—The proof can be divided into two separate cases.

**Case I.**—If all the locally convex spaces $E_i$, $F_j$ are Hausdorff spaces
and their inductive limits $E$ and $F$ are also Hausdorff spaces then the proof
is trivial on the ground that the maps

$$u_i \times v_j : E_i \times F_j \to E \times F$$

are continuous. Hence, the topology $\tau$ is finer than $\varepsilon$ and consequently
$\varepsilon$-equicontinuity of $M$ implies its $\tau$-equicontinuity.

Let us remark that an inductive limit of locally convex Hausdorff
spaces need not be Hausdorff.

**Case II.**—Let the spaces $E_i$ and $F_j$ be not necessarily Hausdorff.

A neighbourhood base in $E$ is given by sets of the form

$$\bigcup u_i (V_i),$$

where for each $i$, $V_i$ is a basic neighbourhood in $E_i$.

Similarly, a neighbourhood base in $F$ is given by sets of the form

$$\bigcup v_j (W_j),$$

where for each $j$, $W_j$ is a basic neighbourhood in $F_j$.

A base of neighbourhoods in $E \times F$ is formed by sets of the form

$$\bigcup u_i \times v_j (V_i^0 \times W_j^0).$$
Some New Results on $\varepsilon$-Tensor Products of Locally Convex Spaces

A base of neighbourhoods in $E \times F$ is formed by sets of the form

$$\left( (\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \right)^0.$$  

Now let $M$ be $\varepsilon$-equicontinuous. Then

$$M \subseteq \left( (\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \right)^{00}$$  

for some choice of $V_i$'s and $W_j$'s.

We shall prove that for such a choice of $V_i$'s and $W_j$'s,

$$M \subseteq (\Gamma \cup i \ X v_j (V_i^0 X W_j^0)^0)^0$$  

which will imply the $\tau$-equicontinuity of $M$.

It is sufficient to prove that

$$\left( (\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \right)^{00} \subseteq \left( \Gamma \cup i, j \ X v_j (V_i^0 X W_j^0)^0 \right)^0.$$  

Hence applying the theorem of bipolars it is sufficient to prove that

$$\Gamma \left( (\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \right) \subseteq (\Gamma \cup i, j \ X v_j (V_i^0 X W_j^0)^0)^0.$$  

Now since the right hand set in " $\subseteq$ " in (4) is weakly closed and absolutely convex, it is sufficient to prove that

$$(\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \subseteq (\Gamma \cup i, j \ X v_j (V_i^0 X W_j^0)^0)^0.$$  

Again since

$$(\Gamma \cup i \ (V_i))^0 \subseteq u_i^{-1} (V_i^0),$$

$$(\Gamma \cup j \ (W_j))^0 \subseteq v_j^{-1} (W_j^0);$$

therefore we have

$$(\Gamma \cup i \ (V_i))^0 \times (\Gamma \cup j \ (W_j))^0 \subseteq u_i^{-1} (V_i^0) \times v_j^{-1} (W_j^0).$$
Hence for proving (5) it is sufficient to prove that

\[ u_i^{-1} (V_i^0) \times v_j^{-1} (W_j^0) \subseteq (\bigcup_{i, j} u_i \times v_j (V_i^0 \times W_j^0)^{\circ \circ}). \quad (6) \]

Now let

\[ z = u_i^{-1} (a_i) \times v_j^{-1} (b_j) \]

be any element of the left hand set of "\( \subseteq \)" in (6),

where

\[ a_i \in V_i^0, \quad b_j \in W_j^0. \]

Any element of

\[ \bigcup_{i, j} u_i \times v_j (V_i^0 \times W_j^0)^{\circ \circ} \]

is of the form

\[ w = \sum_{i, j} a_{ij} u_i \times v_j (w_{ij}), \]

where

\[ \sum_{i, j} |a_{ij}| \leq 1, \quad a_{ij} = 0 \]

except for finitely many \( i \) and \( j \), and \( w_{ij} \in (V_i^0 \times W_j^0)^{\circ \circ} \).

Then

\[ |<z, w>| \]

\[ = |<u_i^{-1} (a_i) \times v_j^{-1} (b_j), \sum_{i, j} a_{ij} u_i \times v_j (w_{ij})>| \]

\[ = |\sum_{i, j} a_{ij} <u_i^{-1} (a_i) \times v_j^{-1} (b_j), u_i \times v_j (w_{ij})>| \]

therefore

\[ |<z, w>| \]

\[ = |\sum_{i, j} a_{ij} <a_i \times b_j, (u_i^{-1} \times v_j^{-1})' (u_i \times v_j) (w_{ij})>| \]

\[ = |\sum_{i, j} a_{ij} <a_i \times b_j, w_{ij}>| \]
Some New Results on e-Tensor Products of Locally Convex Spaces

\[ \leq \sum_{i,j} |a_{ij}| |< a_i X b_j, w_{ij}| \]

\[ \leq \sum_{i,j} |a_{ij}| \]

\[ \leq 1. \]

Hence

\[ z \in (\bigcup_{i,j} u_i X v_j (V_i^0 X W_j^0)^0). \]

Thus we have established the inclusion \( \subseteq \) in (6).

This completes the proof of the proposition.

**Proposition 2.**—Let \( E \) be a locally convex space which is the inductive limit of a family \( (E_i) \) of its locally convex subspaces by the natural injections \( u_i: E_i \hookrightarrow E \).

Let \( F \) be a locally convex space which is the inductive limit of a family \( (F_j) \) of its locally convex subspaces by the natural injections \( j: vF_j \hookrightarrow F \).

Let \( \tau \) be the finest convex topology on \( E X F \) for which the injections

\[ u_i X v_j: E_i X F_j \hookrightarrow E X F. \]

are continuous. Let \( \tau_{i,j} \) be the topology induced on \( E_i X F_j \) by the inductive limit topology \( \tau \) of \( E X F \) and let \( \epsilon_{ij} \) be the topology induced on \( E_i X F_j \) by the e-tensor product topology of \( E X F \).

Then both \( \tau_{ij} \) and \( \epsilon_{ij} \) are weaker than the e-tensor product topology of \( E_i X F_j \).

**Proof.**—Any \( \tau_{ij} \) neighbourhood is the restriction of some \( \tau \)-neighbourhood to \( E_i X F_j \). Now any \( \tau \)-neighbourhood is of the form

\[ \bigcup_{i,j} u_i X v_j (V_i^0 X W_j^0)^0 = \bigcup_{i,j} (V_i^0 X W_j^0)^0, \]

where \( V_i \) and \( W_j \) are neighbourhoods in \( E_i \) and \( F_j \) respectively, belonging to fundamental systems of neighbourhoods in \( E_i \) and \( F_j \) respectively.

Hence any \( \tau_{ij} \)-neighbourhood is of the form

\[ (\bigcup_{i,j} (V_i^0 X W_j^0)^0) \cap (E_i X F_j). \] (1)
Then the following $\epsilon$-neighbourhood in $E_i \times F_j$, namely

$$(V_i^0 \times W_j^0)^0 = (V_i^0 \times W_j^0)^0 \cap (E_i \times F_j)$$

$$\subseteq (\bigcup_{i,j} (V_i^0 \times W_j^0)^0) \cap (E_i \times F_j),$$

for

$$(V_i^0 \times W_j^0)^0 \subseteq \bigcup_{i,j} (V_i^0 \times W_j^0)^0$$

for each $i, j$.

Hence $\tau_{ij}$ is weaker than the $\epsilon$-tensor product topology of $E_i \times X F_j$.

Again, any $\epsilon_{ij}$ neighbourhood is of the form

$$((\bigcup_{i} u_i (V_i))^0 \times (\bigcup_{j} v_j (W_j))^0 \cap (E_i \times F_j)$$

$$= ((\bigcup_{i} V_i)^0 \times (\bigcup_{j} W_j)^0 \cap (E_i \times F_j)$$

(2)

where $V_i$ and $W_j$ belongs to fundamental systems of neighbourhoods in $E_i$ and $F_j$ respectively for each $i, j$.

Now since

$$\bigcup_{i} V_i \supseteq V_i, \quad \bigcup_{j} W_j \supseteq W_j;$$

therefore

$$(\bigcup_{i} V_i)^0 \subseteq V_i^0, \quad (\bigcup_{j} W_j)^0 \subseteq W_j^0.$$

hence finally

$$(V_i^0 \times W_j^0)^0 \subseteq ((\bigcup_{i} V_i)^0 \times (\bigcup_{j} W_j)^0)^0.$$

Therefore the $\epsilon$-neighbourhood $(V_i^0 \times W_j^0)^0$ in $E_i \times X F_j$ is contained in the $\epsilon_{ij}$-neighbourhood defined in (2).

Thus $\epsilon_{ij}$ is also weaker than the $\epsilon$-tensor product topology of $E_i \times X F_j$.

**Proposition 3.**—Let $E$ and $F$ be two Frechet spaces. Then the space $J(E \times F)$ of integral bilinear forms on $E \times F$ is identical with the space of integral bilinear forms on a product of products of two suitable sequences of Banach spaces.

**Proof.**—Let $(p_n), (q_n)$ be sequences of seminorms which define the topology of $E$ and $F$ respectively.
For each $n$, let
\[ M_n = \{ x \in E : p_n(x) = 0 \}; \]
\[ R_n = \{ y \in F : q_n(y) = 0 \}. \]

For each $n$, let $E_n = E/M_n$, $F_n = F/R_n$ be the quotient spaces of $E$ and $F$ respectively module the subspaces $M_n$ and $R_n$.

Let each $E_n$ and $F_n$ be equipped with the quotient norms $\tilde{p}_n$ and $\tilde{q}_n$ respectively. For each $x \in E$, let $x_n$ be its canonical image in $E_n$ and for each $y \in F$, let $y_n$ be its canonical image in $F_n$.

Let $\tilde{E}_n$, $\tilde{F}_n$ denote the completion of $E_n$, $F_n$ respectively. Then the map $u : x \rightarrow (x_n)_{n \in \mathbb{N}}$ is clearly an isomorphism of $E$ into $\prod_{n \in \mathbb{N}} \tilde{E}_n$.

Similarly, the map $v : y \rightarrow (y_n)_{n \in \mathbb{N}}$ is an isomorphism of $F$ into $\prod_{n \in \mathbb{N}} \tilde{F}_n$.

We first prove that $u \times v$ is an isomorphism of $E \times F$ into $(\pi \tilde{E}_n) \times (\pi \tilde{F}_n)$. It is easy to verify that $u \times v$ is an injective linear map. We now prove that $u \times v$ is continuous. For this we identify $E \times F$ with $\mathcal{B}(E_\delta \times F_\delta)$ and $(\pi \tilde{E}_n) \times (\pi \tilde{F}_n)$ with $\mathcal{B}((\Sigma E_n)_\delta \times (\Sigma F_n)_\delta)$. Then $u \times v$ is easily seen to be the mapping which assigns to every continuous bilinear form
\[ (x', y') \rightarrow \Phi(x', y') \text{ on } E'_\delta \times F'_\delta \]
the following continuous bilinear form on $(\Sigma E_n)_\delta \times (\Sigma F_n)_\delta$:
\[ (z', w') \rightarrow \Phi(u'(z'), v'(w')) \]
(1)
where
\[ z' \in \Sigma E_n', w' \in \Sigma F_n'. \]

If $A$ and $B$ denote any equicontinuous subsets of $\Sigma E_n'$ and $\Sigma F_n'$ respectively, then $u'(A)$ and $v'(B)$ are equicontinuous subsets of $E'$ and $F'$ respectively. Hence if the absolute value of $\Phi$ is $\leq 1$ on $u'(A) \times v'(B)$, the absolute value of the form (1) is $\leq 1$ on $A \times B$. Hence $u \times v$ is continuous.

Next we prove that $(u \times v)^{-1}$ is continuous on $(u \times v)(E \times F)$. For this let $U$ and $V$ be any absolutely convex neighbourhoods of origin in $E \times F$. A4
and F respectively. We choose neighbourhoods $U_1, V_1$ of origin in $\pi \tilde{E}_n$ and $\pi \tilde{F}_n$ respectively such that
\[
U_1 \cap u(E) \subset u(U), \quad V_1 \cap v(F) \subset v(V). \tag{2}
\]
Taking the polars of both sides in (2) we get
\[
u^{-1}(U^0) \subset U_1^0 + \nu^{-1}\{0\},
\]
\[
u^{-1}(V^0) \subset V_1^0 + \nu^{-1}\{0\}. \tag{3}
\]
From (3) we get
\[
U^0 \subset u'(U_1^0), \quad V^0 \subset v'(V_1^0). \tag{4}
\]
Hence it follows that, if the bilinear form in (1) converges to zero uniformly on $U_1^0 \times V_1^0$, then $\Phi$ must converge to zero uniformly on $U^0 \times V^0$.

Hence finally $u \times v$ is an isomorphism of
\[
E \times F \text{ into } (\pi \tilde{E}_n) \times (\pi \tilde{F}_n).
\]
Hence extending by continuity, we have $u \tilde{X} v$ as an isomorphism of $E \tilde{X}_\epsilon F$ into $(\pi \tilde{E}_n) \tilde{X}_\epsilon (\pi \tilde{F}_n)$.

Now $E \tilde{X}_\epsilon F$ and $(\pi \tilde{E}_n) \tilde{X}_\epsilon (\pi \tilde{F}_n)$ are Frechet spaces. Hence by the theorem of "Dieudonne-Schwartz," $(u \tilde{X} v)'$ is also an "isomorphism into" for the weak topology and
\[
(u \tilde{X} v)' \text{ into } ((\pi \tilde{E}_n) \tilde{X}_\epsilon (\pi \tilde{F}_n))'.
\]
is closed in $(E \tilde{X}_\epsilon F)'$ for the weak topology.

Hence by a "corollary of the Hahn-Banach theorem", $E \tilde{X}_\epsilon F$ can be embedded in $(\pi \tilde{E}_n) \tilde{X}_\epsilon (\pi \tilde{F}_n)$ as a dense subspace.

Hence finally we have
\[
(E \tilde{X}_\epsilon F)' = ((\pi \tilde{E}_n) \tilde{X}_\epsilon (\pi \tilde{F}_n))',
\]
Some New Results on \( e \)-Tensor Products of Locally Convex Spaces

and therefore

\[
J(E \times F) = J\left(\left(\pi \tilde{E}_n\right) \times \left(\pi \tilde{F}_n\right)\right)
\]

This completes the proof of the proposition.

REFERENCES