# DIFFRACTION OF LIGHT BY SUPERPOSED PARALLEL SUPERSONIC WAVES, BEING HARMONICS OF THE SAME FUNDAMENTAL. SOLUTION OF THE SYSTEM OF DIFFERENCE-DIFFERENTIAL EQUATIONS FOR THE AMPLITUDES

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ABSTRACT

Starting from the general system of difference-differential equations for the amplitudes of the diffracted beams of light, given by Mertens, and using the method of Kuliasko, Mertens and Leroy for the diffraction of light by one supersonic wave, it is possible to reduce the solution of the system of difference-differential equations, to the solution of a partial differential equation. In this way it is possible to calculate the intensities of the order $n$ and $-n$, as a series expansion in $\rho$. Here we only considered terms up to $\rho^2$. It was also possible to verify the general symmetry properties for the intensities studied by Leroy and Mertens.

## 1. INTRODUCTION

THE theory of the diffraction of light by two superposed parallel supersonic waves, the frequency of one being a multiple of the frequency of the other, has been investigated by Ramachandra Rao[1] and Murty,[2] using the methods of Raman and Nath's[3] preliminary theory. Employing the reasoning of Raman and Nath's[4] generalized theory, Mertens,[5] Zankel and Hiedeman[6] extended those calculations, starting from the wave equation for the electric field of light. Other theories, mainly based on Raman and Nath's preliminary theory were developed by Bergmann and Fues,[7] Nagendra Nath[8] and Nagabhushana Rao[9] for sound waves with incommensurable frequencies and no phase-difference. Mertens[10] established a general system of difference-differential equations, taking the phase-differences into account. He solved this system in the case of large wavelengths of the ultrasonics. A study of the general symmetry properties has also been performed by

Leroy and Mertens,[11] who also studied the special case, $n_1 = 1$, $n_2 = 3$, using a method of series expansion.

In the present paper we shall study the diffraction of light by two superposed parallel ultrasonics having frequencies $\nu_1^*$ and $\nu_2^*$, the ratio of which being equal to the ratio of two incommensurable integers $n_1$ and $n_2$, different from one. The method we use here has first been established by Kuliasko, Mertens and Leroy[12] for the diffraction of light by one supersonic wave. We suppose ultrasonics with large wavelengths and calculate in this way the intensities as a series expansion in $\rho = \lambda^2/\mu_0\mu_1\Lambda^2$.

## 1. Establishment of a Partial Differential Equation

As usual we consider a parallel beam of monochromatic light of frequency $\nu$ and wavelength in vacuum $\lambda$, passing through a liquid column, disturbed by two superposed parallel progressive supersonic waves; the ratio of the frequencies of the sound waves is $\nu_1/\nu_2 = n_1/n_2$, $n_1$ and $n_2$ being simple incommensurable integers. The direction of the incident light beam is normal to the direction of the sound waves. Putting the $z$-axis along the direction of the incident light wave, the $x$-axis along the direction of the sound wave, we know that the amplitudes of the diffracted beams of light are given as the solutions of the system of difference-differential equations.[10]

$$2\frac{d\psi_n}{d\zeta} - (\psi_{n-n_1} - \psi_{n+n_1}) - a\,(\psi_{n-n_2}\,e^{-i\delta} - \psi_{n+n_2}\,e^{i\delta}) = i\rho n^2\,\psi_n \quad (1)$$

with

$$\zeta = \frac{2\pi\mu_1 z}{\lambda}\;;\; a = \frac{\mu_2}{\mu_1}\;;\; \delta = n_1\delta_2 - n_2\delta_1 \quad \text{and} \quad \delta_i = 2\pi\Delta_i$$

$$\rho = \frac{\lambda^2}{\mu_0\mu_1\Lambda^2} \quad \text{and} \quad \Lambda = n_1\lambda_1^* = n_2\lambda_2^*\;;$$

The boundary conditions are

$$\psi_n(0) = \delta_{n0}. \quad (2)$$

If we consider the amplitudes $\psi_n(\zeta)$ as the coefficients of the Laurent expansion of an unknown generating function $G(\zeta, \eta)$, which we suppose to be holomorphic in an annular region with centre 0 in the complex $\eta$-plane, we have

$$G(\zeta, \eta) = \sum_{n=-\infty}^{+\infty} \psi_n(\zeta)\,\eta^n \quad (3)$$

the coefficients $\psi_n(\zeta)$ are then given by

$$\psi_n(\zeta) = \frac{1}{2\pi i} \oint \frac{G(\eta, \zeta)}{\eta^{n+1}} d\eta, \tag{4}$$

whereby the contour is any closed path within the annular region, encircling the origin $O$ once in a counterclockwise sense. Multiplying both sides of Equation (1) by $\eta^n$, and summing over all $n$ we find

$$2 \frac{\partial G}{\partial \zeta} - \left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) G - a\left(\eta^{n_1} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right) G$$

$$= i\rho \left(\eta^2 \frac{\partial^2 G}{\partial \eta^2} + \eta \frac{\partial G}{\partial \eta}\right). \tag{5}$$

This equation is a second order partial differential equation. The boundary condition, obtained from (2) and (3) reads

$$G(0, \eta) = 1. \tag{6}$$

The problem is thus reduced to the integration of the partial differential equation (5).

## 3. Solution of the Partial Differential Equation, by Means of a Series Expansion in $\rho$.

Developing the function $G(\zeta, \eta)$ in a series in $\rho$,

$$G(\zeta, \eta) = \sum_{n=0}^{\infty} (i\rho)^n G_n(\zeta, \eta) \tag{7}$$

and substituting it into the equation (5) we find after equating the coefficients of $(i\rho)^n$ on both sides, the following system of differential equations

$$2 \frac{\partial G_n}{\partial \zeta} - \left(\zeta^{n_1} - \frac{1}{\zeta^{n_1}}\right) G_n - a\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right) G_n$$

$$= \eta^2 \frac{\partial^2 G_{n-1}}{\partial \eta^2} + \eta \frac{\partial G_{n-1}}{\partial \eta} \quad (n = 1, 2, \ldots) \tag{8 a}$$

$$2 \frac{\partial G_0}{\partial \zeta} - \left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) G_0 - a\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right) G_0 = 0 \tag{8 b}$$

with

$$G_n(0, \eta) = \delta_{n,0}, \tag{9}$$

the latter condition following from Equation (6).

Integrating (8 *b*) gives

$$G_0 = \exp. \frac{\zeta}{2}\left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) \cdot \exp. \frac{\zeta a}{2}\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2} e^{-i\delta}}\right)$$

or

$$G_0 = \sum_{p,q=-\infty}^{+\infty} J_p(\zeta)\, \eta^{pn_1} \,.\, J_q(a\zeta) \cdot \eta^{qn_2} e^{-iq\delta}. \tag{10}$$

This term $G_0$ corresponds with the special case, $\rho = 0$, and the corresponding amplitude $\psi_n$ is then given by

$$\psi_n = \sum_{p,q=-\infty}^{+\infty} J_p(\zeta) \cdot J_q(a\zeta)\, e^{-iq\delta}, \text{ with } n_1 p + n_2 q = n.$$

This result was also found by Mertens.[10]

The second term of the expansion (7) can be calculated by putting $n = 1$ in Equation (8 *a*) and substituting $G_0$ by the expression (10). So we have

$$2\frac{\partial G_1}{\partial \zeta} - \left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) G_1 - a\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right) G_1$$
$$= \left\{\frac{\zeta}{2}\left[n_1^2\left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) + a n_2^2\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right)\right]\right.$$
$$+ \frac{\zeta^2}{4}\left[n_1^2\left(\eta^{2n_1} + 2 + \frac{1}{\eta^{2n_1}}\right) + a^2 n_2^2\left(\eta^{2n_2} e^{-2i\delta} + 2\right.\right.$$
$$\left. + \frac{1}{\eta^{2n_2}} e^{2i\delta}\right) + 2a n_1 n_2\left(\eta^{n_1+n_2} e^{-i\delta} + \frac{1}{\eta^{n_1+n_2}} e^{i\delta} + \eta^{n_2-n_1} e^{-i\delta}\right.$$
$$\left.\left.\left. + \frac{1}{\eta^{n_2-n_1}} e^{i\delta}\right)\right]\right\} \sum_{p,q=-\infty}^{+\infty} J_p(\zeta) \cdot J_q(a\zeta)\, \eta^{pn_1+qn_2} e^{-iq\delta}. \tag{11}$$

Taking into account the condition (9) for $n = 1$; we obtain after the integration of Equation (11)

$$G_1 = \left\{\frac{\zeta^2}{8}\left[n_1^2\left(\eta^{n_1} - \frac{1}{\eta^{n_1}}\right) + a n_2^2\left(\eta^{n_2} e^{-i\delta} - \frac{1}{\eta^{n_2}} e^{i\delta}\right)\right]\right.$$
$$+ \frac{\zeta^3}{24}\left[n_1^2\left(\eta^{2n_1} + 2 + \frac{1}{\eta^{2n_1}}\right) + a^2 n_2^2\left(\eta^{2n_2} e^{-2i\delta} + 2\right.\right.$$

$$+ \frac{1}{\eta^{2n_2}} e^{2i\delta}\Big) + 2an_1n_2 (\eta^{n_1+n_2} e^{-i\delta} + \eta^{n_2-n_1} e^{-i\delta} + \eta^{n_1-n_2} e^{i\delta}$$

$$+ \eta^{-n_1-n_2} e^{i\delta})\Big]\Big\} \sum_{p,\,q=-\infty}^{+\infty} J_p(\zeta) \cdot J_q(a\zeta)\, \eta^{pn_1+qn_2} e^{-iq\delta}. \qquad (12)$$

In the same way we calculate $G_2$,

$$G_2 = \frac{1}{192} \sum_{p,\,q=-\infty}^{+\infty} J_p(\zeta) \cdot J_q(a\zeta)\, \eta^{pn_1+qn_2} e^{-iq\delta}$$

$$\times \Big[ \zeta^3 \,[4n_1^4 (\eta^{n_1} - \eta^{-n_1}) + 4an_2^4 (\eta^{n_2} e^{-i\delta} - \eta^{-n_2} e^{i\delta})]$$

$$+ \frac{\zeta^4}{2} \{4n_1^4 (17\eta^{2n_1} + 17\eta^{-2n_1} + 6) + 20a^2 n_2^4 (\eta^{2n_2} e^{-2i\delta}$$

$$+ \eta^{-2n_2} e^{2i\delta}) + 10an_1n_2 (n_1^2 + n_2^2) [(\eta^{n_1+n_2} + \eta^{n_2-n_1}) e^{-i\delta}$$

$$+ (\eta^{-n_1-n_2} + \eta^{n_1-n_2}) e^{i\delta} + 17an_1^2 n_2^2 (\eta^{n_1+n_2} - \eta^{n_2-n_1}) e^{-i\delta}$$

$$+ (\eta^{-n_1-n_2} - \eta^{-n_2+n_1}) e^{i\delta}]\} + \frac{\zeta^5}{5} \{13n_1^4 (\eta^{3n_1} + \eta^{n_1} - \eta^{-n_1} - \eta^{-3n_1})$$

$$+ 18a^2 n_1n_2^3 [(\eta^{2n_2+n_1} + \eta^{2n_2-n_1}) e^{-2i\delta} - (\eta^{-2n_2-n_1} + \eta^{-2n_2+n_1}) e^{2i\delta}]$$

$$+ 26an_2n_1^3 [(\eta^{2n_1+n_2} - \eta^{-2n_1+n_2}) e^{-i\delta} - (\eta^{-2n_1-n_2} - \eta^{2n_1-n_2}) e^{i\delta}]$$

$$+ 15an_1^2 n_2^2 [(\eta^{2n_1+n_2} + \eta^{-2n_1+n_2} + 2\eta^{n_2}) e^{-i\delta} - (\eta^{-2n_1-n_2}$$

$$+ \eta^{2n_1+n_2} + 2\eta^{-n_2}) e^{i\delta}] + a^2 n_1^2 n_2^2 [13 (\eta^{n_1+2n_2} - \eta^{-n_1+2n_2}) e^{-2i\delta}$$

$$+ 18 (\eta^{n_1} - \eta^{-n_1}) - 13 (\eta^{-n_1-2n_2} - \eta^{n_1-2n_2}) e^{2i\delta}]$$

$$+ 2a^2 n_1n_2^3 [4 (\eta^{n_1+2n_2} - \eta^{-n_1+2n_2}) e^{-2i\delta} + (-\eta^{-n_1} + \eta^{n_1})$$

$$- 4 (\eta^{-n_1-2n_2} - \eta^{n_1-2n_2}) e^{2i\delta}] + 15a^3 n_2^4 (\eta^{3n_2} e^{-3i\delta} - \eta^{-3n_2} e^{3i\delta}$$

$$+ \eta^{n_2} e^{-i\delta} - \eta^{-n_2} e^{i\delta})\} + \frac{\zeta^6}{6} \{n_1^4 (\eta^{4n_1} + 4\eta^{2n_1} + 6$$

$$+ 4\eta^{-2n_1} + \eta^{-4n_1}) + an_1^2 n_2^2 [(\eta^{2n_2+2n_1} + 2\eta^{2n_2} + \eta^{2n_2-2n_1}) e^{-2i\delta}$$

$$+ 2 (\eta^{2n_1} + 2 + \eta^{-2n_1}) + (\eta^{-2n_2+2n_1} + 2\eta^{-2n_2} + \eta^{-2n_2-2n_1}) e^{2i\delta}]$$

$$+ 4an_1^3 n_2 [(\eta^{3n_1+n_2} + 3\eta^{n_1+n_2} + 3\eta^{-n_1+n_2} + \eta^{n_2-3n_1}) e^{-i\delta}$$

$$+ (\eta^{-3n_1-n_2} + 3\eta^{-n_1-n_2} + 3\eta^{n_1-n_2} + \eta^{-n_2+3n_1})\, e^{i\delta}]$$

$$+ 3a^2 n_1^2 n_2^2 [(\eta^{2n_1+2n_2} + 2\eta^{2n_2} + \eta^{2n_2-2n_1})\, e^{-2i\delta}$$

$$+ (\eta^{-2n_1-2n_2} + 2\eta^{-2n_2} + \eta^{2n_1-2n_2})\, e^{2i\delta} + 2\,(\eta^{-2n_2} + 2 + \eta^{2n_2})]$$

$$+ 4a^3 n_1 n_2^3 [(\eta^{n_1+3n_2} + \eta^{3n_2-n_1})\, e^{-3i\delta} + (\eta^{-n_1-3n_2} + \eta^{n_1-3n_2})\, e^{3i\delta}$$

$$+ 3\,(\eta^{-n_1+n_2} + \eta^{n_2+n_1})\, e^{-i\delta} + 3\,(\eta^{n_1-n_2} + \eta^{-n_1-n_2})\, e^{i\delta}]\}\Big]. \quad (13)$$

## 4. Calculation of the Amplitudes of the Diffracted Light Beams

In the case $\rho \ll 1$, we may neglect terms of higher order than the second one in $\rho$, so that we have approximately

$$G(\zeta, \eta) = G_0 + i\rho G_1 - \rho^2 G_2. \quad (14)$$

Substituting $G_0$, $G_1$ and $G_2$ by their expressions respectively given by (10), (12) and (13), it is easy to verify that we will find, for the function $G$, a sum of terms of the following form

Term of

$$G = T_G = \sum_{p,\, q=-\infty}^{+\infty} f(\zeta, n_1, n_2) \cdot J_p(\zeta) \cdot J_q(\alpha\zeta).$$
$$\times\, \eta^{(p\pm s)\, n_1 + (q\pm r)\, n_2}\, e^{-i\,(q\pm r)\,\delta}. \quad (15)$$

Taking into account (4) and calculating in this way the amplitude, each term of $\psi_n$ will thus been of the form

$$T_{\psi_n} = \frac{1}{2\pi i} \sum_{p,\, q=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, J_p(\zeta)\, J_q(\alpha\zeta)\, e^{-i\,(q\pm r)\,\delta}$$
$$\oint \eta^{(p\pm s)\, n_1 + (q\pm r)\, n_2 - n - 1}\, d\eta. \quad (16)$$

But as we know

$$\frac{1}{2\pi i} \oint \eta^{(p\pm s)\, n_1 + (q\pm r)\, n_2 - n - 1}\, d\eta = \delta_{n,\,(p\pm s)\, n_1 + (q\pm r)\, n_2},$$

thus (16) may be written as

$$T_{\psi_n} = \sum_{p,\, q=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, J_p(\zeta)\, J_q(\alpha\zeta)\, e^{-i\,(q\pm r)\,\delta}\, \delta_{n,\,(p\pm s)\, n_1 + (q\pm r)\, n_2};$$

or

$$T_{\psi_n} = \sum_{p,\, q=-\infty}^{\infty} f(\zeta, n_1, n_2)\, J_{p \mp s}(\zeta)\, J_{q \mp r}(a\zeta)\, e^{-iq\delta}\, \delta_{n,\, pn_1 + qn_2}. \tag{17}$$

So that this expression takes the form

$$T_{\psi_n} = \sum_{p,\, q=-\infty}^{\infty} f(\zeta, n_1, n_2)\, J_{p \mp s}(\zeta)\, J_{q \mp r}(a\zeta)\, e^{-iq\delta}$$

with

$$pn_1 + qn_2 = n. \tag{18}$$

Now it is possible, and seems of great advantage, to substitute the double summation with the indices $p$ and $q$ by one summation with the index $j$, that goes through the row of all integers from $-\infty$ to $+\infty$, and performing the following substitution

$$\begin{aligned} p &= a + jn_2 \\ q &= b - jn_1 \end{aligned} \quad \text{with} \quad an_1 + bn_2 = n. \tag{19}$$

Each term of the amplitude $\psi_n$ will thus been of the form

$$\sum_{j=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, e^{-i(b-jn_1)\delta}\, J_{a+jn_2 \mp s}(\zeta)\, J_{b-jn_1 \mp r}(a\zeta). \tag{20}$$

In this way we find for $\psi_n$

$$\psi_n = \sum_{j=-\infty}^{+\infty} e^{-i(b-jn_1)\delta} \Big[ J_{a+jn_2} \cdot J_{b-jn_1} + i\rho \Big\{ \frac{\zeta^2}{4} (n_1^2 J'_{a+jn_2} \cdot J_{b-jn_1}$$

$$+ an_2^2 J_{a+jn_2} \cdot J'_{b-jn_1}) + \frac{\zeta^3}{24} [n_1^2 (J_{a+jn_2-2} + 2J_{a+jn_2}$$

$$+ J_{a+jn_2+2}) J_{b-jn_1} + a^2 n_2^2 (J_{b-jn_1-2} + 2J_{b-jn_1} + J_{b-jn_1+2})$$

$$\times J_{a+jn_2} + an_1n_2 (J_{a+jn_2-1} + J_{a+jn_2+1})(J_{b-jn_1-1} + J_{b-jn_1+1})] \Big\}$$

$$- \frac{1}{192} \rho^2 \Big\{ 8\zeta^3 (n_1^4 J'_{a+jn_2} \cdot J_{b-jn_1} + an_2^4 J'_{b-jn_1} \cdot J_{a+jn_2})$$

$$+ \frac{\zeta^4}{2} [4n_1^4 (17J_{a+jn_2-2} + 6J_{a+jn_2} + 17J_{a+jn_2+2}) J_{b-jn_1}$$

$$+ 4a^2 n_2^4 (5J_{b-jn_1-2} + 3J_{b-jn_1} + 5J_{b-jn_1+2}) J_{a+jn_2}$$
$$+ 10an_1n_2 (n_1^2 + n_2^2) (J_{a+jn_2-1} + J_{a+jn_2+1}) (J_{b-jn_1-1}$$
$$+ J_{b-jn_1+1}) + 34an_1^2 n_2^2 J'_{a+jn_2} \cdot J'_{b-jn_1}]$$
$$+ \frac{\zeta^5}{5} [13n_1^4 (J_{a+jn_2-3} + J_{a+jn_2-1} - J_{a+jn_2+1} - J_{a+jn_2+3}) J_{b-jn_1}$$
$$+ 18a^2 n_1^3 n_2 (J_{a+jn_2-1} - J_{a+jn_2+1}) (J_{b-jn_1-2} - J_{b-jn_1+2})$$
$$+ 26an_1^3 n_2 (J_{a+jn_2-2} - J_{a+jn_2+2}) (J_{b-jn_1-1} + J_{b-jn_1+1})$$
$$+ 30an_1^2 n_2^2 (J_{a+jn_2-2} + 2J_{a+jn_2} + J_{a+jn_2+2}) J'_{b-jn_1}$$
$$+ 2a^2 n_1^2 n_2^2 J'_{a+jn_2} (13J_{b-jn_1-2} + 18J_{b-jn_1} + 13J_{b-jn_1+2})$$
$$+ 4a^2 n_1n_2^3 J'_{a+jn_2} (4J_{b-jn_1-2} + J_{b-jn_1} + 4J_{b-jn_1+2})$$
$$+ a^3 n_2^4 15 (J_{b-jn_1-3} - J_{b-jn_1+3} + 2J'_{b-jn_1}) J_{a+jn_2}]$$
$$+ \frac{\zeta^6}{6} [n_1^4 (J_{a+jn_2-4} + 4J_{a+jn_2-2} + 6J_{a+jn_2} + 4J_{a+jn_2+2}$$
$$+ J_{a+jn_2+4}) \cdot J_{b-jn_1} + (1 + 3a) n_1^2 n_2^2 (J_{a+jn_2-2} + 2J_{a+jn_2}$$
$$+ J_{a+jn_2+2}) \cdot (J_{b-jn_1-2} + 2J_{b-jn_1} + J_{b-jn_1+2})$$
$$+ 4an_1^3 n_2 (J_{a+jn_2-3} + 3J_{a+jn_2-1} + 3J_{a+jn_2+1} + J_{a+jn_2+3})$$
$$\times (J_{b-jn_1-1} + J_{b-jn_1+1}) + 4a^3 n_1n_2^3 (J_{a+jn_2-1} + J_{a+jn_2+1})$$
$$\times (J_{b-jn_1-3} + 3J_{b-jn_1-1} + 3J_{b-jn_1+1} + J_{b-jn_1+3})]\}\Big] . \qquad (21)$$

(In the above formula, the Bessel functions with an index beginning with $a$ have the argument $\zeta$, those whose index starts with $b$ have the argument $a\zeta$).

Or, in general form we have, if we substitute in (21) the constant term by $A_j$, the coefficient of $i\rho$ by $B_j$ and the coefficient of $-\rho^2$ by $C_j$

$$\psi_n \sum_{j=-\infty}^{+\infty} e^{-i(b-jn_1)\delta} (A_j + i\rho B_j - \rho^2 C_j). \qquad (22)$$

## 5. Intensities

If we neglect terms in $\rho$ higher than the second order, and taking into account the expression (22) we find for the intensity of the order $n$

$$I_n = \psi_n\psi_n^* = \sum_{j,\,k=-\infty}^{+\infty} A_kA_j \cos (k-j) n_1\delta + 2\rho A_kB_j \sin (k-j) n_1\delta$$
$$- \rho^2 (2A_kC_j - B_kB_j) \cos (k-j) n_1\delta. \qquad (23)$$

Before calculating $I_{-n}$, we consider the general term of the amplitude $\psi_{-n}$, by substituting $n$ by $-n$ in the relation (17)

$$T_{-n} = \sum_{p,\,q=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, J_{p\mp s}(\zeta)\, J_{p\mp r}(\alpha\zeta)\, e^{-iq\delta}\, \delta_{-n,\,pn_1+qn_2} \tag{24}$$

or

$$T_{\psi-n} = \sum_{p,\,q=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, J_{p\mp s}(\zeta)\, J_{q\mp r}(\alpha\zeta)\, e^{-iq\delta}, \tag{25}$$

with the condition $pn_1 + qn_2 = -n$,

or if

$$p = -a + jn_2$$
$$q = -b + jn_1$$

we find

$$T_{\psi-n} = \sum_{j=-\infty}^{+\infty} f(\zeta, n_1, n_2)\, e^{-i(b-jn_1)\delta}\, J_{-a+jn_2\mp s}(\zeta)\, J_{-b-jn_1\mp r}(\alpha\zeta). \tag{26}$$

Taking into account that $J_{-n}(x) = (-1)^n J_n(x)$ there results

$$T_{\psi-n} = \sum_{j=-\infty}^{+\infty} (-1)^{a+b+r+s+j(n_1-n_2)} f(\zeta, n_1, n_2)\, e^{-i(b+jn_1)}\, J_{a+jn_2\mp s}(\zeta)$$
$$J_{b-jn_1\mp r}(\alpha\zeta). \tag{27}$$

From (27) we conclude that each term of $\psi_{-n}$ with sommation index $j$ can be calculated from the corresponding term in $\psi_n$ also with sommation index $j$, on the following way

(1) we substitute $e^{-i(b-jn_1)\delta}$ by $e^{-(b+jn_1)\delta}$

(2) we multiply by $(-1)^{a+b+r+s+j(n_1-n_2)}$.

In this way it is possible to write $\psi_{-n}$ as follows

$$\psi_{-n} = \sum_{j=-\infty}^{+\infty} e^{-i(b+jn_1)\delta} (-1)^{(a+b)+(r+s)+j(n_1-n_2)} (A_j + i\rho B_j - \rho^2 C_j), \tag{28}$$

and the intensity $I_{-n}$ becomes

$$I_{-n} = \sum_{j,\,k=-\infty}^{+\infty} (-1)^{(k+j)(n_1-n_2)} (A_k A_j \cos(k-j)\, n\delta - 2\rho\, A_k B_j \sin$$
$$(k-j)\, n_1\delta - \rho^2 (2C_k A_j - B_k B_j) \cos(k-j)\, n_1\delta]. \tag{29}$$

The symmetry properties follow from the relations (23) and (29). We find that the spectrum will be symmetric if

$$(-1)^{(k+j)(n_1-n_2)} \cos (k-j) n_1\delta = \cos (k-j) n_1\delta \tag{30}$$

and

$$(-1)^{(k+j)(n_1-n_2)} \sin (k-j) n_1\delta = -\sin (k-j) n_1\delta. \tag{31}$$

Those are fullfilled if and only if,

(1) $n_1 - n_2$ is even and $\delta$ = even multiple of $\pi/2n_1$

(2) $n_1 - n_2$ is odd and $\delta$ = odd multiple of $\pi/2n_1$,

which corresponds with the general symmetry properties obtained by Leroy and Mertens.[11]

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