ENERGY CONSIDERATIONS OF THE FLOW BEHIND A PLANE HYDROMAGNETIC SHOCK

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ABSTRACT

Similarity solutions describing the flow behind a plane hydromagnetic shock propagating with a constant velocity into a uniform ideal gas at rest in the presence of a transverse magnetic field are obtained. The gas is assumed to be infinitely electrically conducting, inviscid and non-heat conducting. The gain in the total energy of the flow between the shock and the inner expanding surface is assumed to be time-dependent. The variations of the percentages of the magnetic, internal and kinetic energies with the strength of the shock are studied. It is shown that there exists two values of the strength of the shock at which equipartition of the internal and kinetic energies of the flow between the shock and the inner expanding surface can occur.

1. INTRODUCTION

AN instantaneous release of energy gives rise to a blast wave headed by a shock front. As the shock front propagates and encloses more of the ambient gas, the flow behind the shock front gains more internal energy. But in case of intense explosions where the ambient gas pressure is negligible in comparison to the pressure behind the shock, this gain of the energy of the flow may be neglected. With this assumption, Taylor[1] obtained the similarity solutions describing the initial stages of an intense explosion. The similarity method adopted by Taylor has become the basis for many investigations[2, 3] in this direction. By taking into account the energy gain by the flow, Rogers[4] obtained a more generalized class of similarity solutions for the flows behind strong shocks. Rogers assumed the energy gain with time t in the form

$$E = E_0 t^s, \tag{1}$$

where E is the total energy of the flow between the shock and the inner expanding surface, E_0 and s are constants. In blast wave 80 per cent of the total energy of the flow is in the form of internal energy and only 20 per cent in the form of kinetic energy. In case of a strong plane shock propagating with constant velocity into a uniform ideal gas at rest, Rogers[4] showed that there exists an equipartition of the kinetic and internal energies of the flow (for $s = 1$).

In this paper the similarity solutions describing the flow behind a plane hydromagnetic shock propagating with constant velocity into a uniform ideal gas at rest are obtained. The gas is assumed to be infinitely electrically conducting, inviscid and non-heat conducting. The initial magnetic field is assumed to be uniform and is in the direction perpendicular to the flow. Then it follows that the gain in the total energy (sum of kinetic, internal and magnetic energies) of the flow is directly proportional to the time [*i.e.*, $s = 1$, in equation (1)]. The variations of the percentages of kinetic, internal and magnetic energies of the flow with the strength of the shock are studied. It is shown that there exist two values of the strength of shock at which equipartition of the internal and kinetic energies of the flow between the shock and the inner expanding surface can occur.

One of the authors* has studied the propagation of plane and cylindrical hydromagnetic shocks into a uniform gas at rest in the presence of transverse magnetic field by taking the energy gain into account.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

Let p, ρ, u, H, U, X (t) and γ denote the pressure, density, velocity, transverse magnetic field, shock velocity, co-ordinate to specify the shock position with $X(0) = 0$, and adiabatic index of the ideal gas respectively. Then the equation of continuity, momentum, entropy and magnetic induction are given in the non-dimensional form as

$$(f - z)\, h' + h f' = 0, \tag{2}$$

$$(f - z) f' + \frac{X}{U^2}\frac{dU}{dt} f + \frac{g'}{\gamma h} + \frac{\beta^2}{h}\frac{a_0^2}{U^2} G G' = 0, \tag{3}$$

$$(f - z)\, g' + \gamma g f' + 2\frac{dU}{dt}\frac{X}{U^2} g = 0, \tag{4}$$

$$(f - z)\, G' + G f' = 0, \tag{5}$$

* P. Chaturani, Shock Waves in Magnetogasdynamics (Ph.D. Thesis, I.I.T., Bombay, 1968, unpublished).

where

$$\left.\begin{array}{l} f(z) = u/\mathrm{U},\ g(z) = p/\rho_0\mathrm{U}^2,\ h(z) = \rho/\rho_0, \\ \mathrm{G}(z) = \mathrm{H}/\mathrm{H}_0,\ a_0^2 = \gamma p_0/\rho_0,\ b_0^2 = \mu_e \mathrm{H}_0^2/\rho_0, \\ \beta^2 = b_0^2/a_0^2,\ z = x/\mathrm{X},\ \mathrm{U} = d\mathrm{X}/dt \end{array}\right\}. \tag{6}$$

Here the subscript 0 denotes the value of the quantity ahead of the shock and a prime stands for the differentiation with respect to the argument z. The shock conditions (at $z = 1$) serve as the moving boundary conditions to the system of differential equations (2) to (5) and are given in the non-dimensional form[5] as

$$h(1) = \xi,\ \mathrm{G}(1) = \xi,\ f(1) = \frac{\xi - 1}{\xi}, \tag{7}$$

$$g(1) = \frac{a_0^2}{\mathrm{U}^2}\left[1 + \frac{2\gamma(\xi - 1)\left\{1 + \frac{\gamma - 1}{4}\beta^2(\xi - 1)^2\right\}}{(\gamma + 1) - (\gamma - 1)\xi}\right] \tag{8}$$

$$\frac{\mathrm{U}^2}{a_0^2} = 2\xi\,\frac{\left[1 + \frac{\beta^2}{2}\{\gamma + (2 - \gamma)\xi\}\right]}{\{(\gamma + 1) - (\gamma - 1)\xi\}}.$$

The total energy of the flow between the shock and the inner expanding surface of unit cross-section is given by

$$\mathrm{E} = \mathrm{U}^2\rho_0\mathrm{X}\int_{z_0}^{1}\left\{\frac{1}{2}hf^2 + \frac{g}{\gamma(\gamma - 1)} + \frac{\mu_e\mathrm{H}_0}{\rho_0\mathrm{U}^2}\mathrm{G}^2\right\}dz, \tag{9}$$

where z_0 is the co-ordinate of the inner expanding surface which is given by the kinematic condition on it, $z_0 = f(z_0)$. It can be easily shown,[4] from equations (1) and (9), that $s = 1$ when the inner surface expands uniformly.

3. Results and Discussion

When U is constant, the system of equations (2) to (5) with boundary conditions (7) and (8) has the following solutions

$$h(z) = \mathrm{G}(z) = \xi, \tag{10}$$

$$f(z) = \frac{\xi - 1}{\xi}, \tag{11}$$

$$g(z) = \frac{a_0^2}{U^2}\left[1 + \frac{2\gamma(\xi - 1)\left\{1 + \frac{\gamma - 1}{4}\beta^2(\xi - 1)^2\right\}}{(\gamma + 1) - (\gamma - 1)\xi}\right]. \tag{12}$$

Substitution of (10), (11) and (12) in (8) yields

$$\mathrm{E} = \frac{\gamma p_0 \mathrm{X}}{\xi}\left[\frac{\left\{1 + \frac{\beta^2}{2}(\gamma + (2 - \gamma)\xi)\right\}(\xi - 1)^2}{(\gamma + 1) - (\gamma - 1)\xi} + \frac{\beta^2}{2}\xi^2 + \frac{1}{\gamma(\gamma - 1)}\left\{1 + \frac{2\gamma(\xi - 1)\left(1 + \beta^2\frac{(\gamma - 1)}{4}(\xi - 1)^2\right)}{(\gamma + 1) - (\gamma - 1)\xi}\right\}\right]. \tag{13}$$

The percentages of magnetic, internal and kinetic energies of the flow are, respectively, given by

$$\frac{100\mathrm{X}}{\mathrm{E}}\int_{z_0}^{1}\frac{\mu_e \mathrm{H}^2}{2}\,dz = \frac{100}{2\mathrm{E}}\gamma p_0 \mathrm{X}\xi\beta^2, \tag{14}$$

$$\frac{100\mathrm{X}}{\mathrm{E}}\int_{z_0}^{1}\frac{p}{(\gamma - 1)}\,dz = \frac{100\mathrm{X}p_0}{\mathrm{E}(\gamma - 1)}\left[1 + \frac{2\gamma(\xi - 1)\left(1 + \frac{\gamma - 1}{4}\beta^2(\xi - 1)^2\right)}{(\gamma + 1) - (\gamma - 1)\xi}\right], \tag{15}$$

$$\frac{100\mathrm{X}}{\mathrm{E}}\int_{z_s}^{1}\frac{1}{2}\rho u^2\,dz = \frac{100\mathrm{X}}{\mathrm{E}}\rho_0 \mathrm{U}^2\frac{(\xi - 1)^2}{\xi^2}. \tag{16}$$

The variation of the percentages of magnetic, internal and kinetic energies with ξ are shown through Figs. (1) to (3). Figure 1 shows that as the strength of shock (ξ) increases, the percentage of magnetic energy increases and reaches a maximum value and then starts decreasing and attains the zero value when $\xi = \gamma + 1/\gamma - 1$, *i.e.*, shock is infinitely strong. From

Fig. 2, it is clear that with the increase of ξ the percentage of internal energy decreases and reaches a minimum value; then starts increasing and becomes equal to the percentage of kinetic energy, when shock becomes infinitely strong. It may be noticed that no such *minimum* is observed in the case of gasdynamic shocks. Similarly, an interesting result regarding the kinetic energy may also be noticed from Fig. 3. The percentage of kinetic energy increases with the increase of ξ and attains a maximum value; then it starts decreasing and becomes equal to the percentage of internal

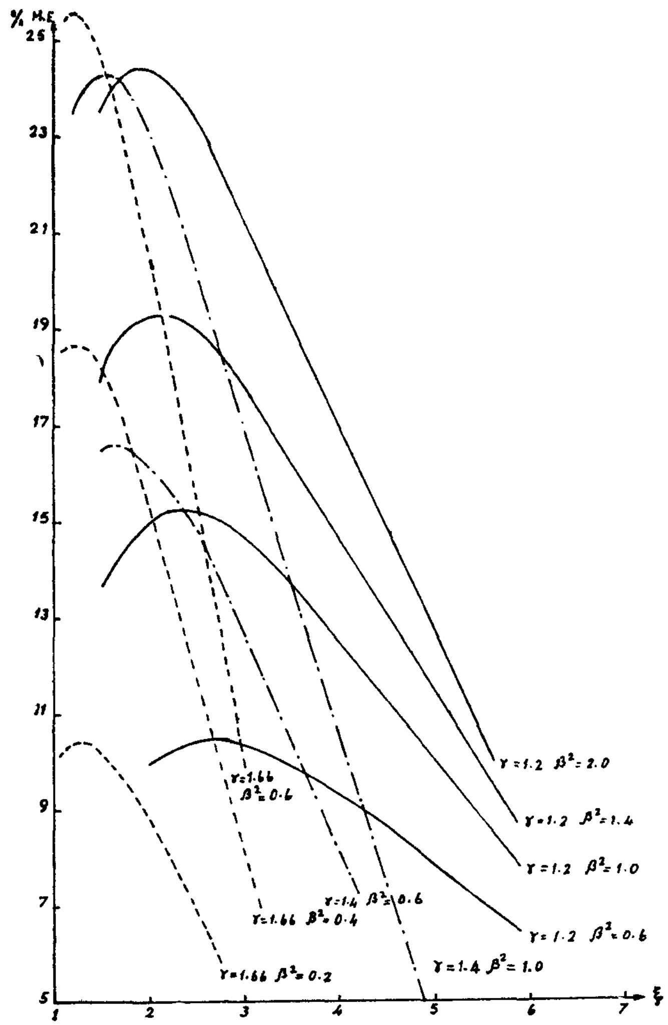

FIG. 1. The variation with ξ (shock strength) of the percentage of magnetic energy in the total energy of the flow.

energy when shock becomes infinitely strong whereas in gasdynamic shocks no such maximum is observed.

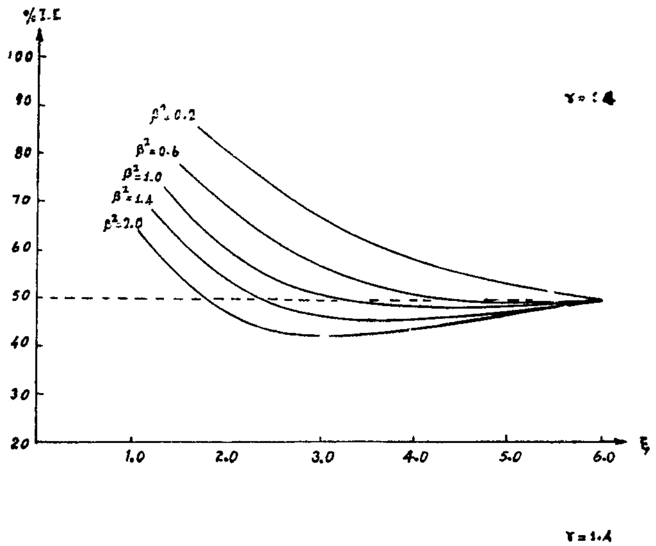

FIG. 2. The variation with ξ of the percentage of internal energy in the total energy of the flow.

A careful examination of the percentages of the internal and kinetic energy curves in figures reveals an interesting information regarding equipartition of energies. It shows that *some of these two types of curves may intersect* at a point below the 50 per cent line. At this point of intersection the percentage of the kinetic and internal energies are equal. Hence the value of ξ corresponding to the point of intersection may also be considered as the value of the shock strength at which equipartition of the kinetic and internal energies takes place. The equipartition of kinetic and internal energies also occurs when shock is infinitely strong. Thus, it is concluded that there are two values of the strength of shock, in magnetogasdynamics at which the equipartition of kinetic and internal energies can occur. This is in contrast to the ordinary gasdynamics where only one value of the strength of shock ($\xi = \gamma + 1/\gamma - 1$) exists at which the equipartition of the energies can take place.

It is of interest to support this numerical observation by an analytic proof. To proceed with we first equate the kinetic and internal energies

$$\frac{1}{2} hf^2 = \frac{g}{\gamma (\gamma - 1)} \,. \tag{17}$$

Now by substituting h, f and g from (10) and (11) in (17), we get

$$\frac{\beta^2}{2}(1-\gamma)\,\xi^3 + \left\{1 + \frac{\beta^2}{2}(3\gamma - 1)\right\}\xi^2$$

$$- \left\{\frac{\beta^2}{2}(3\gamma + 1) + \frac{2\gamma^2 - \gamma + 1}{(\gamma - 1)}\right\}\xi + (\gamma + 1)\left(\frac{\beta^2}{2} + \frac{1}{\gamma}\right) = 0. \tag{18}$$

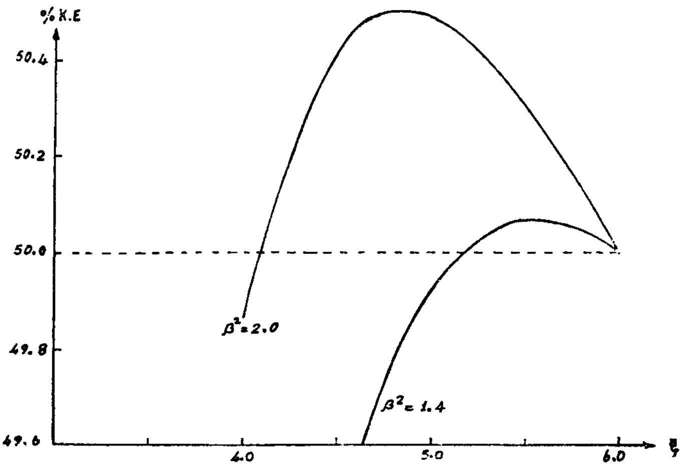

FIG. 3. The variation of the percentage of the kinetic energy in the enegy (sum of the K.E and I.E) of the flow with ξ.

The equation (18) is a cubic equation in ξ and has three real roots ξ_1, ξ_2 and ξ_3 which are given by

$$\xi_1 = \frac{\gamma + 1}{\gamma - 1}, \tag{19}$$

$$\xi_2 = \frac{1 + (\gamma - 1)\beta^2 + \sqrt{1 + \frac{2}{\gamma}\beta^2(\gamma - 1)}}{(\gamma - 1)\beta^2}, \tag{20}$$

$$\xi_3 = \frac{1 + (\gamma - 1)\beta^2 - \sqrt{1 + \frac{2}{\gamma}\beta^2(\gamma - 1)}}{(\gamma - 1)\beta^2}, \tag{21}$$

For a compressive shock ξ must be always greater than unity so the root given by ξ_3 which is less than unity, for all values of β^2, is rejected. *Thus the existence of the two possible values of ξ at which equipartition of internal* *and kinetic energies can occur is established.* From (20), it is evident that ξ_2 depends on β^2 whereas ξ_1 is independent of β^2. Also the range for ξ_2 is $1 \leq \xi_2 \leq \xi_1$. The values of ξ_2 will be in this range if and only if $\beta^2 > (1 \cdot 5 - 0 \cdot 5/\gamma)$. If this condition is satisfied then and only then there exist two values of ξ given by ξ_1 and ξ_2 at which the equipartition of the internal and kinetic energies can occur. If $\beta^2 < (1 \cdot 5 - 0 \cdot 5/\gamma)$ then, similar to gasdynamics, there will be only one value of ξ given by ξ_1 at which equipartition occurs.

Besides this type of equipartition there can be other types of equipartition of energies such as an equipartition of magnetic and kinetic energy, internal energy = magnetic energy + kinetic energy, etc. Proceeding in the same manner as given above, we can obtain the corresponding algebraic equation for ξ. For example if there is an equipartition of kinetic and magnetic energy, then the equation for ξ is

$$\frac{\beta^2}{2} \xi^3 + \left\{1 + \frac{\beta^2}{2}(2\gamma - 5)\right\} \xi^2 + \left\{\frac{\beta^2}{2}(2 - 3\gamma) - 2\right\} \xi + 1 + \frac{\gamma}{2} \beta^2 = 0. \tag{23}$$

Thus for any required equipartition the corresponding equation for ξ can be obtained.

In conclusion it may be said that in case of plane magnetogasdynamic shocks propagating with uniform velocity, there exist two values of the strength of shock at which equipartition of kinetic and internal energies can occur. In ordinary gasdynamic shocks there exists only one value of the strength of shock at which the equipartition of the energies can occur. Besides this, in contrast to ordinary gasdynamic shocks where only one type of equipartition of the energies can occur, in magnetogasdynamic shocks various types of equipartition of the energies can occur.

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