POLYNOMIAL BASIS FOR THE IRREDUCIBLE REPRESENTATIONS OF U₆ IN THE CHAIN U₆ ⊇ R₆

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ABSTRACT

Explicit polynomial basis for the Irreducible Representation (IR) \([K_1, K_2, K_3]\) of \(U_6\) containing an IR \((\lambda_1, \lambda_2, \lambda_3)\) of \(R_6\) is obtained. These polynomials are useful in calculating the nuclear energy levels in the \(2s-1d\) shell.

I. INTRODUCTION

RECENTLY, studies in Many-Body problems are being carried out by Moshinsky et al.¹, ², ³ using Group Theoretical techniques. But these calculations can be very much simplified by using Littlewood's rules to construct bases for unitary groups in terms of physically interesting subgroups of all types. Here, in this paper, we illustrate this straightforward procedure to obtain an explicit polynomial basis for the IR \([K_1, K_2, K_3]\) of \(U_6\) containing an IR \((\lambda_1, \lambda_2, \lambda_3)\) of \(R_6\). We give the Highest Weight Polynomials (HWPs), of IRS \((\lambda_1, \lambda_2, \lambda_3)\) of \(R_6\) contained in an IR \([K_1, K_2, K_3]\) of \(U_6\), only. The full basis can be obtained by applying the Lowering operators for \(R_6\)⁴, ⁵ on these HWPs. The polynomials in the chain \(U_6 ⊇ R_3\) are important and they are given by Bargmann and Moshinsky.⁶ But these polynomials can be obtained very elegantly by the above-mentioned technique.

II (a). THE HWP OF THE IR \((\lambda_1, \lambda_2, \lambda_3)\) OF \(R_6\) CONTAINED IN AN IR \([K_1, K_2, K_3]\) OF \(U_6\)

The Basis for the IR (BIR)⁷ \([K_1, K_2, \ldots, K_{2l+1}]\) of \(U_{2l+1}\) are homogeneous polynomials in boson creation operators \(a_{\mu s}\) and are solutions of the equations

\[
C_{ss'}P = K_sP, \quad (s = 1, 2, \ldots, 2l + 1),
\]

and

\[
C_{ss'}P = 0, \quad (s < s' = 2, 3, \ldots, 2l + 1),
\]

\[\text{(II.1)}\]
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where

\[
C_{ss'} = \sum_{\mu=1}^{2l+1} a_{\mu s} a_{s' \mu}, \quad (s, s' = 1, 2, \ldots, 2l + 1).
\]

The infinitesimal generators of \( U_6 \) are

\[
C_{\mu \mu'} = \sum_{s=1}^{6} a_{\mu s} a_{s \mu'}, \quad (\mu, \mu' = 1, 2, \ldots, 6).
\]

The infinitesimal generators of \( R_6 \) can be chosen as

\[
\begin{align*}
H_1 &= C_1^1 - C_6^6, \\
H_2 &= C_2^2 - C_5^5, \\
H_3 &= C_3^3 - C_4^4, \\
E_1 &= C_1^2 - C_5^6, \\
E_2 &= C_2^3 - C_4^5, \\
E_3 &= C_2^4 - C_3^5, \\
E_4 &= C_1^2 - C_4^6, \\
E_5 &= C_1^4 - C_3^6, \\
E_6 &= C_1^5 - C_2^6, \\
E_{-1} &= C_2^1 - C_6^5, \\
E_{-2} &= C_3^2 - C_5^4, \\
E_{-3} &= C_4^2 - C_6^3, \\
E_{-4} &= C_3^1 - C_6^4, \\
E_{-5} &= C_4^1 - C_5^3, \\
E_{-6} &= C_5^1 - C_2^6.
\end{align*}
\]

A simple system of roots are \((1, -1, 0), (0, 1, -1)\) and \((0, 1, 1)\). The generators corresponding to this simple system of roots are respectively \(E_1, E_2\) and \(E_3\).

The HWP of the IR \((\lambda_1, \lambda_2, \lambda_3)\) of \( R_6 \) contained in an IR \([K_1, K_2, K_3]\) of \( U_6 \) is a solution of the equations

\[
\begin{align*}
C_{11} P &= K_1 P, \\
C_{22} P &= K_2 P, \\
C_{33} P &= K_3 P, \\
C_{44} P &= 0, \\
C_{55} P &= 0, \\
C_{66} P &= 0; \\
C_{ss'} P &= 0; \quad (s < s' = 2, 3, 4, 5, 6), \\
H_1 P &= \lambda_1 P, \\
H_2 P &= \lambda_2 P, \\
H_3 P &= \lambda_3 P,
\end{align*}
\]

and

\[
\begin{align*}
E_1 P &= 0, \\
E_2 P &= 0, \\
E_3 P &= 0.
\end{align*}
\]

II (b). The IR \([K_1, K_2, K_3, K_4, K_5]\) of \( U_6 \) containing a given IR \((\lambda_1, \lambda_2, \lambda_3)\) of \( R_6 \)

The IR \((\lambda) = (\lambda_1, \lambda_2, \lambda_3)\) of \( R_6 \) contained in an IR \([K] = [K_1, K_2, K_3, K_4, K_5]\) of \( U_6 \) can be obtained by using a theorem of Littlewood,\( ^8 \) viz.,

\[
[K] = \sum_{(\delta)} g_{\delta \lambda k} (\lambda),
\]

where

\[
(\delta) (\lambda) = \sum_{[K]} g_{\delta \lambda k} [K],
\]
and (δ) is a partition into even parts. The product (δ) (λ) and hence the coefficients $g_{δ\lambda k}$ are determined by Littlewood's rules. We must also note that an IR $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_3 > 0$, of $R_6$ splits into two non-equivalent IRs of the same dimension with respect to $R_6^9$. The spaces in the Young diagram corresponding to $(\lambda_1, \lambda_2, \lambda_3)$ are denoted by ‘x’s. The spaces in the 1st, 2nd and 3rd rows of $(\delta) = (2p, 2q, 2r)$ are respectively denoted by ‘a’s, ‘b’s and ‘c’s. The Young diagram corresponding to the IR $[K_1, K_2, K_3]$ of $U_6$ then has the form

$$
\begin{align*}
\begin{array}{c}
\text{x} \text{x} \text{x} \ldots \text{x} \text{x} \text{x} \\
\text{x} \text{x} \text{x} \ldots \\
\text{x} \text{x} \text{x} \\
\end{array}
\begin{array}{c}
\text{a} \text{a} \text{a} \\
\text{a} \text{a} \\
\text{a} \\
\end{array}
\begin{array}{c}
\text{a} \text{a} \text{a} \\
\text{b} \text{b} \text{b} \\
\text{c} \text{c} \ldots \\
\end{array}
\end{align*}
$$

(D)

where $x_{11} + x_{12} + x_{13} = 2p$, $x_{22} + x_{23} = 2q$, $x_{33} = 2r$.

The Littlewood’s rules give us the following inequalities:

$$\begin{align*}
&x_{11} \geq x_{22} \geq x_{33}, \quad x_{11} + x_{12} \geq x_{22} + x_{23}, \\
&\lambda_1 \geq \lambda_2 + x_{12}, \quad \lambda_2 \geq \lambda_3 + x_{13}, \quad \lambda_2 + x_{12} \geq \lambda_3 + x_{13} + x_{23}.
\end{align*}$$

(II.5)

Due to the fact that the lengths of the 1st, 2nd and 3rd rows of (D) are equal to $K_1$, $K_2$ and $K_3$ respectively we get

$$\begin{align*}
\lambda_1 + x_{11} = K_1, \quad \lambda_2 + x_{12} + x_{22} = K_2, \quad \lambda_3 + x_{13} + x_{23} + x_{33} = K_3.
\end{align*}$$

(II.6)

Any three of the five quantities $x_{12}$, $x_{13}$, $x_{22}$, $x_{23}$, $x_{33}$ can be chosen as parameters, which subject to the inequalities (II.5) and equations (II.6), distinguish between the different irreducible sub-spaces corresponding to the same $(\lambda_1, \lambda_2, \lambda_3)$ of $R_6$ occurring in the reduction of the IR $[K_1, K_2, K_3]$ of $U_6$.

II (c). Explicit Construction of the HWP of the IR $(\lambda_1, \lambda_2, \lambda_3)$ of $R_6$ Contained in an IR $[K_1, K_2, K_3]$ of $U_6$

Divide the diagram (D) columnwise into what may be called Elementary Permissible Diagrams (EPDs). A diagram is said to be permissible if the added symbols ‘a’, ‘b’ and ‘c’ of the partition (δ) occurring in it are each even in number, and the total set of added symbols ‘a’, ‘b’ and ‘c’ is in lattice
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order. A permissible diagram is an EPD if it cannot be split into two or more permissible diagrams.

\[
\begin{align*}
x x & \quad \quad \quad \quad \quad \quad \quad \quad a a \\
x x & \text{ is an EPD whereas } \quad \quad \quad \quad \quad \quad \quad \quad a a \\
a a & \quad \quad \quad \quad \quad \quad \quad \quad a a
\end{align*}
\]

since it can be split into two permissible diagrams \(x x\) and \(a a\).

The splitting of the diagram \((D)\) into its constituent EPDs is not unique. But by adopting a suitable convention, namely, that each column of \((D)\) combines with the nearest column to its right as we go from left to right so as to form an EPD, the splitting is made unique.

For example

\[
\begin{align*}
x x & \quad \quad \quad \quad \quad \quad \quad \quad a a \\
x x & \quad \quad \quad \quad \quad \quad \quad \quad a a \\
a a & \quad \quad \quad \quad \quad \quad \quad \quad a a
\end{align*}
\]

but not into the EPDs \(x a\) and \(x a\)

\[
\begin{align*}
x a & \quad \quad \quad \quad \quad \quad \quad \quad a a \\
a & \quad \quad \quad \quad \quad \quad \quad \quad a
\end{align*}
\]

After splitting the diagram \((D)\) into EPDs we associate a unique polynomial with each EPD as follows. We take the Young pattern of the EPD and fill the spaces with '1', '2', '3', '4', '5', and '6' satisfying the following conditions. (i) The number of '1's minus the number of '6's is equal to \(\lambda_1\), the number of 'x's in the 1st row of the EPD; (ii) the number of '2's minus the number of '5's is equal to \(\lambda_2\), the number of 'x's in the 2nd row of the EPD and (iii) the number of '3's minus the number of '4's is equal to \(\lambda_3\), the number of 'x's in the 3rd row of the EPD. In general we get several such filled Young patterns. With each column of \(r\) rows of a filled Young pattern we associate a polynomial

\[
\sum_{\sigma} (-1)^{\sigma} a_{s_1,1} a_{s_2,2} \ldots a_{s_r,r}
\]

where the summation is over all permutations of the indices \(s_1, s_2, \ldots, s_r\) and \(s_1, s_2, \ldots, s_r\) are the numbers occurring in the column respectively in the 1st, 2nd, \ldots, \(r\)-th rows. For the sake of convenience we write \((s_1 s_2 \ldots s_r)\) for \(\sum_{\sigma} (-1)^{\sigma} a_{s_1,1} a_{s_2,2} \ldots a_{s_r,r}\). Then the polynomial associated with the filled Young pattern is \(\Pi (s_1 s_2 \ldots s_r)\), where \(\Pi\)
stands for the product over all the columns of the pattern. The polynomials thus associated with the several filled Young patterns of an EPD satisfy the equations (II.3.a) but need not satisfy the equations (II.3.b). We then take a linear combination of these polynomials so as to satisfy the equations (II.3.b). This linear combination is the polynomial associated with the EPD. In the cases of $U_3 \supset R_3$ and $U_4 \supset R_4$ we get a unique linear combination for each EPD. But in our present case we get two independent linear combinations satisfying the equations (II.3.a) and (II.3.b) for certain EPDs. Even in these cases we associate a unique polynomial with each EPD which is explained in the Appendix at the end of this paper. The polynomial associated with the diagram (D) is the product of the polynomials associated with its constituent EPDs with appropriate frequencies. This product polynomial also satisfies the equations (II.3.a) and (II.3.b) since our operators are differential operators. Thus the HWP of the IR $(\lambda_1, \lambda_2, \lambda_3)$ of $R_6$ contained in an IR $[K_1, K_2, K_3]$ of $U_6$ is obtained.

The EPDs into which the diagram (D) is split are listed below.

$$
D_1 \rightarrow x; \quad D_2 \rightarrow x; \quad D_3 \rightarrow x; \quad D_4 \rightarrow xx; \quad D_5 \rightarrow xx; \\
D_6 \rightarrow xx; \quad D_7 \rightarrow xaa; \quad D_8 \rightarrow xaa; \quad D_9 \rightarrow xaa; \\
D_{10} \rightarrow xx; \quad D_{11} \rightarrow xa; \quad D_{12} \rightarrow xx; \quad D_{13} \rightarrow xx; \\
D_{14} \rightarrow a; \quad D_{25} \rightarrow a; \quad D_{16} \rightarrow a; \\
D_{17} \rightarrow b; \quad D_{18} \rightarrow b; \quad D_{19} \rightarrow b; \\
D_{20} \rightarrow c; \quad D_{21} \rightarrow c; \quad D_{22} \rightarrow c; \\
D_{23} \rightarrow c; \quad D_{24} \rightarrow c; \quad D_{25} \rightarrow c;
$$

Choosing $x_{12}$, $x_{13}$ and $x_{23}$ as parameters the frequencies of the EPDs, in the above order, are

$$
f_1 = \lambda_1 - \lambda_2 - x_{12}; \quad f_2 = \lambda_2 - \lambda_3 - x_{13} - x_{23} + M; \\
f_3 = \lambda_3; \quad f_4 = [x_{13}/2]; \quad f_5 = \text{Min.} (x_{13} - 2f_4; \ x_{12} - M);
$$
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$f_6 = \text{Min.} (x_{13} - 2f_4 - f_5); \quad K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12});$

$f_7 = \text{Min.} \left(\left[(x_{23} - M)/2\right]; \left[(K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12} - f_6)/2\right]\right);$

$f_8 = \text{Min.} (x_{23} - M - 2f_7; \quad M; \quad K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12} - f_6 - 2f_7);$

$f_9 = \text{Min.} (x_{23} - M - 2f_7 - f_8; \quad K_2 - K_3 + \lambda_3 - \lambda_2 + x_{13} - x_{12} + x_{23}; \quad K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12} - f_6 - 2f_7 - f_8);$

$f_{10} = [(x_{12} - M - f_5)/2];$

$f_{11} = \text{Min.} (x_{12} - M - f_5 - 2f_{10}; \quad K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12} - f_6 - 2f_7 - f_5);$

$f_{12} = [(M - f_5)/2];$

$f_{13} = \text{Min.} (M - f_5 - 2f_{12}; \quad K_2 - K_3 + \lambda_3 - \lambda_2 + x_{13} - x_{12} + x_{23});$

$f_{14} = (K_1 - K_2 + \lambda_2 - \lambda_1 + x_{12} - f_6 - 2f_7 - f_6 - f_5 - f_{11})/2;$

$f_{15} = (K_2 - K_3 + \lambda_3 - \lambda_2 + x_{13} - x_{12} + x_{23} - f_5 - f_{13} - f_{14})/2;$

$f_{16} = (K_3 - \lambda_3 - x_{13} - x_{23})/2.$

$M = \text{Min.} (x_{12}; x_{23}). [n]$ stands for the integral part of the number $n.$

The polynomials associated with the EPDs are

$P_1 = (1); \quad P_2 = (12); \quad P_3 = (123); \quad P_3' = (124);$

$P_4 = (123) (124); \quad P_5 = (125) (12) + (123) (14) + (124) (13);$

$P_6 = (126) (1) + (125) (2) + (124) (3) + (123) (4);$

$P_7 = (126)^2 (1)^2 + 2 (125) (126) (1) (2) + 2 (124) (126) (1) (3) + 2 (123) (126) (1) (4) - 2 (123) (124) (1) (6) + (125)^2 (2)^2 + 2 (124) (125) (2) (3) + 2 (123) (125) (2) (4) - 2 (123) (124) (2) (5) + (124)^2 (3)^2 + (123)^2 (4)^2;$

\[ \begin{align*}
P_9 &= (126)(16)(1) + (126)(15)(2) + (125)(16)(2) \\
&
+ (126)(12)(5) - 2(125)(12)(6) + (126)(14)(3) \\
&
&
&
&
&
- (123)(34)(4) + (124)(34)(3); \\
P_{10} &= (12)(16) + (13)(14); \\
P_{11} &= (16)(1) + (15)(2) + (12)(5) + (14)(3) + (13)(4); \\
P_{12} &= (125)^2 + (134)^2 + 2(124)(135) + 2(123)(145); \\
P_{13} &= 2(126)(15) - (125)(16) + 2(136)(14) - (134)(16) \\
&
&
- (125)(34) + 2(124)(35); \\
P_{14} &= (1)(6) + (2)(5) + (3)(4); \\
P_{15} &= (16)^2 + (25)^2 + (34)^2 + 2(15)(26) + 2(12)(56) \\
&
+ 2(14)(36) + 2(13)(46) + 2(24)(35) + 2(23)(45); \\
P_{16} &= (126)(156) + (136)(146) + (125)(256) + 2(135)(246) \\
&
- (134)(256) - (125)(346) + 2(124)(356) + (134)(346) \\
&
+ (235)(245) + (234)(345). \\
\end{align*} \]

The HWPs of the IR \((\lambda_1, \lambda_2, \lambda_3)\) of \(R_6\) contained in an IR \([K_1, K_2, K_3]\) of \(U_6\) are

\[ P = \sum_{i=1}^{16} P_i I^i \quad \text{and} \quad P' = \sum_{i=1}^{16} P_i I^i, \]

where in \(P'\) the third term in the product is \(P_3' = (124)\).

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**References**

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APPENDIX

We will now show how a unique polynomial can be associated with an EPD in those cases where we get two linearly independent polynomial solutions of the equations (II.3.a) and (II.3.b). Let us consider the simple example, namely the EPD.

$$D_7 \rightarrow x \times a \times a$$

$$x \times b \times b.$$ 

In writing down the polynomial associated with this EPD we are actually writing the HWP of the IR (2, 2) of $R_6$ contained in the IR [4, 2, 2] of $U_6$. Since (2, 2) of $R_6$ occurs twice in the reduction of [4, 2, 2] of $U_6$, we are getting the two polynomial solutions. They must be properly associated with the two ways of obtaining (2, 2) of $R_6$ from [4, 2, 2] of $U_6$. The two solutions we get in this case are

$$P_7 \text{ and } (123) (124) (1) (6) + (123) (124) (2) (5) + (123) (124) (3) (4).$$

Let us denote the second solution as $Q$.

(2, 2) of $R_6$ can be obtained from [4, 2, 2] of $U_6$ in two ways, namely, one by removing 4 ‘a’s, as in

$$x \times a \times a \text{ and the other by removing 2 ‘a’s and 2 ‘b’s, as in } x \times a \times a$$

$$x \times \quad x \times$$

$$a \times a, \quad b \times b.$$ 

But the polynomial associated with $x \times a \times a$ is nothing but the product is

$$x \times \quad x \times$$

$$a \times a, \quad a \times a,$$

of the polynomials associated with the two EPDs, $x \times a \times a$

$$x \times \quad x \times$$

$$a \times a, \quad a \times a,$$

into which it splits. This product polynomial can be easily seen to be $Q$. So we associate the polynomial $P_7$ with the other way of obtaining (2, 2) of $R_6$ from [4, 2, 2] of $U_6$, namely $x \times a \times a$

$$x \times \quad x \times$$

$$b \times b,$$

which is our EPD, $D_7$. 

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