FRACTIONAL INTEGRATION AND $\omega_{\mu\nu}$-TRANSFORM

BY D. G. JOSHI

[Department of Mathematics, Holkar Science College, Indore (M.P.)]

Received May 28, 1968

(Communicated by Dr. P. L. Bhatnagar, F.A.Sc.)

ABSTRACT

In this paper a formula connecting self-reciprocal functions of different order in $\omega_{\mu\nu}$-transform are developed by employing fractional integration. Further it is shown that the process can be extended to other Fourier kernels. This is illustrated by considering the transform defined by Bhatnagar.

1. INTRODUCTION

RECENTLY, R. Naraiu\(^7\) has developed a formula connecting functions of different classes of self-reciprocal functions in Hankel transform by employing fractional integration. In this paper we develop a similar formula connecting self-reciprocal functions of different order in $\omega_{\mu\nu}$-transform, the kernel introduced by Watson.\(^9\) Further we will show that process can be extended to Bhatnagar kernel\(^1\) and to other Fourier kernels in general.

2. According to Titchmarsh\(^8\) (page 252) let us say that $f(x) \in A(a, a)$ where $0 < a \leq \pi$, $a < \frac{1}{2}$, if

(i) $f(x)$ is regular function of $x = re^{i\theta}$ for $r > 0$, $|\theta| < a$, and

(ii) $f(x) = 0 (|x|^{-a-\epsilon})$ for small $x$, and

$f(x) = 0 (|x|^{a+1+\epsilon})$ for large $x$, for every positive $\epsilon$ and uniformly for $|\theta| \leq a - \eta < a$.

Let $s = \sigma + it$ be a complex variable. Following Titchmarsh\(^8\) (page 252) the author\(^5\) has established the following result.

A necessary and sufficient condition that a function $f(x) \in A(a, a)$, should be its own K-transform is such that its Mellin transform, $K(s)$ is $O(1)$, $K_1(s)$ is $O(e^{\lambda|t|})$ where

$$K(s) = \frac{K_1(s)}{K_1(1 - s)} \quad (2.1)$$

230
is such that $f(x)$ should be of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_1(s) \psi(s) x^{-s} \, ds$$  \tag{2.2}$$

with

$$\psi(s) = \psi(1-s)$$  \tag{2.3}$$

where $\psi(s)$ and $K_1(s)$ are regular in the strip

$$a < \sigma < 1-a, \quad a < \frac{1}{2}$$  \tag{2.4}$$

$\psi(s)$ is $0(e^{-\lambda-a+\eta})^{it}$, for every positive $\eta$ and uniformly in the strip (2.4) and $c$ is any value of $\sigma$ in the strip (2.4).

In the case of $\omega_{\mu,\nu}$-transform, introduced by Watson the function $f(x)$ assumes the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma\left(\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2}\right) \Gamma\left(\frac{1}{4} + \frac{\nu}{2} + \frac{s}{2}\right) \psi(s) x^{-s} \, ds$$  \tag{2.5}$$

with

$$\psi(s) = \psi(1-s)$$ and is $0(e^{(\pi/2-a+\eta)})^{it}$ for every positive $\eta$ and uniformly in the strip (2.4). $f(x)$ is then said to be $R_{\mu,\nu}$.

Similarly, Bhatnagar\(^1\) has shown that a function $A(a, a)$ self-reciprocal in $\omega_{p_1, p_2, ..., p_m}(x)$ transform denoted by $R_{p_1, p_2, ..., p_m}$ is of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{m\\nu} \Gamma\left(\frac{1}{4} + \frac{\nu_1}{2} + \frac{s}{2}\right) \cdot \Gamma\left(\frac{1}{4} + \frac{\nu_m}{2} + \frac{s}{2}\right) \psi(s) x^{-s} \, ds$$  \tag{2.6}$$

with $\psi(s) = \psi(1-s)$ and is $0(e^{(m\\nu/4-a+\eta)})^{it}$ uniformly in the strip (2.4).

3. Fractional integration and Mellin Transform.—The operators of fractional integration with respect to $x^A$, for $A > 0$, as given by Erde'lyi\(^3\) are as follows

$$\Gamma_{a, \eta}^A f(x) = \frac{1}{\Gamma(a)} x^{-A\eta-Aa} \int_0^\infty (x^A - t^A)^{a-1} t^{A\eta} f(t) \, dt$$  \tag{3.1}$$
232 □ D. G. JOHNS

\[ K^\eta_a f(x) = \frac{1}{I(a)} x^{\eta \eta} \int_0^\infty (t^a - x^a)^{a-1} t^{-\eta a} f(t) \, dt. \quad (3.2) \]

We have\(^a\) (theorem 44, p. 60) that if \( f(x) \in L(0, \infty) \), \( g(x) \in L(0, \infty) \), then \( (f \ast g)(x) \in L(0, \infty) \), where

\[ (f \ast g)(x) = \int_0^\infty \frac{1}{u} f\left(\frac{x}{u}\right) g(u) \, du. \quad (3.3) \]

The set \( L(0, \infty) \) of complex valued functions forms an algebra over the field of complex numbers with usual definitions of addition, scalar multiplication and the convolution (3.3) as the definition of the product.

Let \( U(x) \), be the Heaviside unit function given by

\[ U(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0 \end{cases} \quad (3.4) \]

and

\[ I^n, a, A(x) = \frac{A}{I(a)} (x^a - 1)^{a-1} x^{-\eta - \Lambda a} U(x - 1). \quad (3.5) \]

Bushman\(^2\) has shown that the fractional integral operators (3.1) and (3.2) can be written in the convolution form

\[ I^n, a, A f(x) = (I^n, a, A \ast f) (x) \quad (3.6) \]

\[ I^n, a, A(x) \in L(0, \infty) \text{ for } a > 0, \eta > \frac{1}{A} - 1. \]

Similarly, if \( U(x) \) is as in (3.4) and

\[ K^\eta_a (x) = \frac{A}{I(a)} (1 - x^A)^{a-1} x^{\eta A} U(1 - x) \quad (3.7) \]

then

\[ K^\eta_a f(x) = (K^\eta_a \ast f)(x) \quad (3.8) \]

\[ K^\eta_a (x) \in L(0, \infty) \text{ for } a > 0, \eta > -\frac{1}{A}. \]

The operators \( I \)'s and \( K \)'s associate, and commute if \( A = B \).
**Fractional Integration and \( \omega_{\mu \nu} \)-Transform**

The Mellin Transform \( F(s) \) of \( f(x) \) is given by

\[
M[f(x)] = F(s) = \int_0^{\infty} f(x) x^{s-1} dx. \quad (3.9)
\]

where \( s \) is a complex variable \( (s = \sigma + it) \) and the inverse Mellin Transform of \( F(s) \) is given by

\[
M^{-1}[F(s)] = f(x) = \frac{1}{2\pi i} \oint_{C} F(s) x^{-s} ds. \quad (3.10)
\]

The convolution theorem for Mellin Transform is given by

\[
F(s) G(s) = M[(f*g)(x)] \quad (3.11)
\]

where

\[
s = c + it, \quad x^c f(x) \in L(0, \infty), \quad x^c g(x) \in L(0, \infty)
\]

and

\[
x^c (f*g)(x) \in L(0, \infty).
\]

It has been proved by Bushman\(^2\) that

\[
M[I^{\eta, \alpha, \Lambda}(x)] = \frac{\Gamma(1 + \eta - s/\Lambda)}{\Gamma(1 + \eta + \alpha - s/\Lambda)}, \quad \alpha > 0, \quad \text{Re} \, s < \Lambda (\eta + 1) \quad (3.12)
\]

\[
M[K^{\eta, \alpha, \Lambda}(x)] = \frac{\Gamma(\eta + s/\Lambda)}{\Gamma(\eta + \alpha + s/\Lambda)}, \quad \alpha > 0, \quad \text{Re} \, s > -\Lambda \eta \quad (3.13)
\]

4. **Theorem**: If \( f(x) \in R_{\nu_1, \nu_2} \) as given in (2.5), then

\[
K_{\nu_1, \mu_2, (\nu_1 - \mu_2)/2} f(x) \in R_{\mu_1, \nu_1} \quad (4.1)
\]

and

\[
K_{\nu_1, \mu_2, (\nu_1 - \mu_2)/2} (R_{\mu_1, \nu_1}) \in R_{\mu_1, \mu_1} \quad (4.2)
\]

**Proof.**—As \( f(x) \in R_{\nu_1, \nu_2} \), we have from (2.5)

\[
M[f(x)] = 2^s \Gamma \left( \frac{1}{4} + \frac{\nu_1}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\nu_2}{2} + \frac{s}{2} \right) \psi(s). \quad (4.3)
\]
Using (4.3) and (3.13) in (3.11) we have
\[ M \{[K^\eta, \alpha, 2 \cdot f](x) \} \]
\[ = 2^s \Gamma \left( \frac{\eta + s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\nu_1}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\nu_2}{2} + \frac{s}{2} \right) \psi(s) \]
\[ \Gamma \left( \frac{\eta + \alpha + s}{2} \right) \]
\[ \alpha > 0, \text{ Re } s > - \alpha \eta \quad (4.4) \]

Selecting
\[ \eta = \frac{1}{4} + \frac{\mu_1}{2}, \quad \alpha = \frac{1}{2} (\nu_1 - \mu_1) \]

(4.4) becomes
\[ M \{[K^{\frac{1}{4} + \mu_1/2, (\nu_1 - \mu_1)/2, 2 \cdot f}](x) \} \]
\[ = 2^s \Gamma \left( \frac{1}{4} + \frac{\mu_1}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\nu_2}{2} + \frac{s}{2} \right) \psi(s) \quad \nu_1 > \mu_1 > -1. \quad (4.5) \]

Using (2.5) we conclude that
\[ K^{\frac{1}{4} + \mu_1/2, (\nu_1 - \mu_1)/2} f(x) \in R_{\mu_1, \nu_1}. \]

Repeating this process we have
\[ M \{K^\eta, \alpha, 2 \cdot \{K^{\frac{1}{4} + \mu_1/2, (\nu_1 - \mu_1)/2, 2 \cdot f} \}(x) \} \]
\[ = 2^s \Gamma \left( \frac{\eta + s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\mu_1}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\mu_2}{2} + \frac{s}{2} \right) \psi(s) \quad (4.6) \]

Selecting
\[ \eta = \frac{1}{4} + \frac{\mu_2}{2}, \quad \alpha = \frac{\nu_2 - \mu_2}{2}, \]

(4.6) becomes
\[ M \{K^{\frac{1}{4} + \mu_2/2, (-\mu_2 + \nu_2)/2, 2 \cdot \{K^{\frac{1}{4} + \mu_1/2, (\nu_1 - \mu_1)/2, 2 \cdot f} \}(x) \} \]
\[ = 2^s \Gamma \left( \frac{1}{4} + \frac{\mu_1}{2} + \frac{s}{2} \right) \Gamma \left( \frac{1}{4} + \frac{\mu_2}{2} + \frac{s}{2} \right) \psi(s). \quad \text{Re } \mu_2 > -1, \text{ Re } (\nu_2 - \mu_2) > 0. \]
Fractional Integration and $o_{\mu_\nu}$-Transform

Thus from (2.5) again
\[ [K_{\gamma}^{1+\mu_2/2, (v_2-\mu_2)^2}, \{K_{\gamma}^{1+\mu_3/2, (v_3-\mu_3)^2} f\}(x)] \in R_{\mu_1, \mu_3} \]

As the K-operators commute we also have
\[ [K_{\gamma}^{1+\mu_2/2, (v_2-\mu_2)^2} \{K_{\gamma}^{1+\mu_3/2, (v_3-\mu_3)^2} f\}(x)] \in R_{\mu_1, \mu_3}. \]

Instead of $o_{\mu_\nu}$-transform, we consider the transform defined by Bhatnagar$^1$ and if
\[ K(\mu_1, v_1) = K_{\gamma}^{1+\mu_4/2, (v_4-\mu_4)^2} \]

then the operators
\[ K(\mu_1, v_1) K(\mu_2, v_2), \ldots, K(\mu_m, v_m) \]

will transform a $R_{\mu_1, \nu_1}$ function into $R_{\mu_1, \mu_2, \mu_3, \ldots, \mu_m}$.

The operators $K$ can be permuted in $\prod_{r}$ ways. The process can be employed in the case of Fourier kernels by using (2.2).

5. Example.—Following R. Narain$^6$ it can be shown that a G-function which is self-reciprocal in $o_{\nu_1, \nu_2}$-transform and denoted by the symbol $R_{\nu_1, \nu_2}$ is of the form
\[ f_1(x) = AG_{n+2}^{n+2} \frac{a_1, a_2, \ldots a_n}{n+2} \times \left( \frac{x^2}{4} \frac{1}{4 + \nu_1}, \frac{1}{4 + \nu_2}, \frac{1}{2} - a_1, \frac{1}{2} - a_2, \ldots \frac{1}{2} - a_n \right) \]

where $A$ is some constant. $\text{Re} (v_1) > -1$, $\text{Re} (v_2) > -1$, $\text{Re} (v_1 - 2\alpha_j) > -5/2$, $\text{Re} (v_2 - 2\alpha_j) > -5/2$, $j = 1, 2, \ldots, n$.

Operating with $K(\mu_1, \nu_1)$ on $f(x)$ and evaluating the integral with the help of result$^4$ (page 417) we get
\[ [K(\mu_1, \nu_1) f_1](x) \]

\[ = A G_{n+2}^{n+2} \left( \frac{x^2}{4} \frac{1}{4 + \mu_1}, \frac{1}{4 + \nu_1}, \frac{1}{2} - a_1, \ldots \frac{1}{2} - a_n \right) \]

$\text{Re} \mu_1 > -1$, $\text{Re} \nu_2 > -1$, $\text{Re} (\mu_1 - 2\alpha_j) > -5/2$, \n
$\text{Re} (\nu_2 - 2\alpha_j) > -1, j = 1, 2, \ldots, n$ \hspace{1cm} (5.2)

\[ = f_2(x). \]
which is $R_{\mu_1, \nu_2}$.

Repeating the process on $f_2(x)$ we have
\[
[K (\mu_2, \nu_2) f_2] (x)
\]
\[
= A \, G_{n+2}^{n+2} \left( \frac{x^2}{4} \begin{array}{c} a_1, \ldots, a_n \\ \frac{1}{4} + \frac{\mu_1}{2}, \frac{1}{4} + \frac{\mu_2}{2}, \frac{1}{2} - a_1, \ldots, \frac{1}{2} - a_n \end{array} \right)
\]
\[
\text{Re} \, \mu_1 > -1, \, \text{Re} \, \mu_2 > -1, \, \text{Re} (\mu_1 - 2a_j) > -5/2,
\]
\[
\text{Re} (\mu_2 - 2a_j) > -5/2, \, j = 1, \ldots, n \quad \nu_2 > \mu_2
\]

which is $R_{\mu_1, \nu_2}$.

(5.3)

For the Bhatnagar Transform\(^1\) consider
\[
f_3(x) = A \, G_{n+r}^{n+r} \left( \frac{x^2}{2r} \begin{array}{c} a_1, \ldots, a_n \\ 1 + \frac{\nu_1}{2}, \ldots, 1 + \frac{\nu_r}{2}, \frac{1}{2} - a_1, \ldots, \frac{1}{2} - a_n \end{array} \right),
\]
\[
0 < r < n
\]
\[
\text{Re} \, \nu_m > -1, \, m = 1, \ldots, r, \, \text{Re} (\nu_m - 2a_j) > -5/2,
\]
\[
\text{Re} \, a_j < \frac{1}{2} \, m = 1, \ldots, r, \, j = 1, \ldots, n,
\]
\[
\text{Re} (\nu_m + \nu_p) > -2, \, m = 1, \ldots, r,
\]
\[
j = 1, \ldots, n, \, p = 1, \ldots, r.
\]

which is $R_{\nu_1, \nu_2, \ldots, \nu_r}$.

Operating on $f_3(x)$ by
\[
K (\mu_1, \nu_1) \ldots K (\mu_r, \nu_r)
\]

we get
\[
[K (\mu_1, \nu_1) K (\mu_2, \nu_2), \ldots, K (\mu_r, \nu_r) f_3(x)]
\]
\[
= A \, G_{n+r}^{n+r} \left( \frac{x^2}{2r} \begin{array}{c} a_1, \ldots, a_n \\ \frac{1}{4} + \frac{\mu_1}{2}, \frac{1}{4} + \frac{\mu_2}{2}, \ldots, \frac{1}{4} + \frac{\mu_r}{2}, \frac{1}{2} - a_1, \ldots, \frac{1}{2} - a_n \end{array} \right),
\]
\[
\nu_r > \mu_r
\]
Fractional Integration and $\omega_{\mu r}$-Transform

$\text{Re } \mu_m > -1, \, m = 1, \ldots r, \, \text{Re } (v_m - 2a_j) > -5/2,$
$m = 1, \ldots r, \, j = 1, \ldots n, \, \text{Re } (v_m + v_p) > -2,$
$\text{Re } (\mu_m - 2a_j) > -5/2, \, |\arg x| < (2n + r) \pi/4,$
$m = 1, \ldots r, \, p = 1, \ldots r, \, j = 1, \ldots n.$

which is $R_{\mu_1, \mu_2, \ldots, \mu_r}$ in the Bhatnagar Transform.

My thanks are due to Dr. V. M. Bhise, Shri G. S. Technological Institute, Indore, for his help and guidance during the preparation of this paper. I am also thankful to Dr. S. M. Das Gupta, Principal, Shri G. S. Technological Institute, Indore, for the facilities he gave me.

**REFERENCES**

1. Bhatnagar, K. P. 

2. Bushman, R. G. 

3. Erde'lyi, Arthur 

4. ———— 
   *Tables of Integral Transform*, 1954, 2.

5. Joshi, D. G. 

6. Narain, R. 

7. ———— 

8. Titchmarsh, E. C. 

9. Watson, G. N. 