THREE-DIMENSIONAL PERIODIC BOUNDARY LAYERS

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ABSTRACT

The method of successive approximations used by Eichelbrenner and Aškovíc[1] for the study of unsteady three-dimensional boundary layer flow has been extended to analyse the periodic boundary layers in three dimensions. The analysis, which is valid for oscillations of small amplitude, shows some special features such as "steady streaming" flow in the first-order cross-flow similar to the one that has been predicted and observed by Schlichting for two-dimensional periodic boundary layers.

INTRODUCTION

THE analysis of Eichelbrenner and Aškovič[1] in which the pri :ciple of 'prevalence' has been adapted to study the unsteady three-dimensional boundary layers has been used here to study the periodic boundary layers in three dimensions. This analysis is also an extension of Schlichting's[2] work in two-dimensional periodic boundary layers. The usual method of solving the three-dimensional boundary layer equations of a steady flow past a finite obstacle is based on the following three hypotheses:

Hypothesis 1.—Every stream line inside the boundary layer has a well-determined limiting position on the surface of the obstacle as well as at the edge of the boundary layer. Conversely, at every point of the boundary layer there exists only one streamline. This is true at the wall as well as at the edge.

Hypothesis 2.—In a system of curvilinear co-ordinates based on the direction of the external streamlines (*viz.*, 'Streamline co-ordinates') around an obstacle, we can, in the first approximation, neglect to the order of $R^{-\frac{1}{2}}$ (where R is the Reynolds number) the transverse component of the velocity vector with respect to the longitudinal component. Then the three-dimensional boundary layer equations degenerate into a system of two-dimensional boundary layer equations *plus* an additional equation for the transverse flow. This is the principle of prevalence.

Hypothesis 3.—In a stream tube, the quasi-two-dimensional system for the longitudinal flow can be interpreted as the system corresponding to a local axisymmetric flow.

On this basis the equations have been generalised to the case of the unsteady flow. The above three hypotheses have been adopted to the case of the unsteady three-dimensional boundary layer flow in reference (1). Following the same reasoning in the present problem also it is assumed that

$$\frac{V_e(p;\, t) - V_e(p;\, t_1)}{|V_e(p;\, t)|} \ll 1$$

for $t_1 \leqslant t \leqslant t_2$ where t, represents a characteristic time, say, the period $T = 2\pi/\omega$, ω being the frequency. Further in the streamline co-ordinates, as the cross-flow velocity w has to become zero at the edge of the boundary layer and it is also zero at the wall, it is assumed to be small everywhere compared to the streamwise velocity u. That is $w/u \ll 1$ throughout the boundary layer. The equations are further simplified by assuming the curvature variations also as small.

THE EQUATIONS

We use the curvilinear co-ordinate system (ξ, η, ζ) where $\eta = 0$ represents the general three-dimensional body surface on which ξ and ζ are assumed to be orthogonal and the η-axis is taken to be perpendicular to the surface so that the above system of co-ordinates are 'locally' orthogonal at the surface. The error in such an assumption is negligible as our region of interest is quite close to the body surface according to the Prandtl's boundary layer assumptions. Hence, the boundary layer equations can now be written as

$$\frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{w}{h_2}\frac{\partial u}{\partial \zeta} + v\frac{\partial u}{\partial \eta} + \frac{uw}{h_1 h_2}\frac{\partial h_1}{\partial \xi} - \frac{w^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$= -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \nu\frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial p}{\partial \eta} = 0,$$

$$\frac{\partial w}{\partial t} + \frac{u}{h_1}\frac{\partial w}{\partial \xi} + \frac{w}{h_2}\frac{\partial w}{\partial \zeta} + v\frac{\partial w}{\partial \eta} + \frac{uw}{h_1 h_2}\frac{\partial h_2}{\partial \xi} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \zeta}$$

$$= -\frac{1}{\rho h_2}\frac{\partial p}{\partial \zeta} + \nu\frac{\partial^2 w}{\partial \eta^2},$$

$$\frac{\partial(\rho h_2 u)}{\partial \xi} + \frac{\partial(\rho h_1 w)}{\partial \zeta} + \frac{\partial(\rho h_1 h_2 v)}{\partial \eta} = 0 \qquad (1)$$

where u, w, v are the velocities in the directions ξ, ζ, η and h_1 $(\xi\ \zeta)$, h_2 (ξ, ζ), $h_3(\xi, \zeta) = 1$ are, respectively, the scale factors in these directions. These scale factors are related to the line element dl by

$$dl^2 = h_1^2 d\xi^2 + h_2^2 d\zeta^2 + h_3^2 d\eta^2. \qquad (2)$$

Let the differential line elements in the three directions be ds, dz, and dn so that

$$ds = h_1 d\xi, \quad dz = h_2 d\zeta, \quad dn = d\eta. \qquad (3)$$

Further, as has been done in reference (1), we assume that the transverse velocity component or the cross-flow w is much smaller compared to u. The equations (1) then become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\frac{\partial^2 u}{\partial n^2}, \qquad (4)$$

$$\frac{\partial(h_2 u)}{\partial s} + \frac{\partial(h_2 v)}{\partial n} = 0, \qquad (5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial s} + v\frac{\partial w}{\partial n} - \frac{u^2 \partial h_1}{h_1 \partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial n^2}. \qquad (6)$$

The equations (4) and (5) are independent of (6) and are the same as the equations for the two-dimensional boundary layers. Hence, thay can be solved to obtain u and v which on substitution in (6) give w.

The boundary conditions are

$$u = v = w = 0, \ n = 0; \ u = u_e, \ n = \infty \qquad (7)$$

where $u_e = u_e(s, z, t)$ is the velocity of the unsteady potential flow at the edge of the boundary layer. The pressure gradients in equations (4) and (6) are then given by (since $\partial p/\partial n = 0$)

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} = \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial s}, \tag{8}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{u_e^2}{h_1}\frac{\partial h_1}{\partial z}. \tag{9}$$

Now we are interested in obtaining the solutions of (4), (5) and (6) for a periodic potential flow with zero steady part, *i.e.*, the problem is to solve for u, v, w in the boundary layer when the velocity at the edge of the boundary layer u_e is of the form

$$u_e = U_0(s, z) \cos \omega t$$

or

$$= U_0(s, z)\, e^{i\omega t} \tag{10}$$

with the convention that only the real parts of the complex quantities in question have physical meaning attached to them. With the known outer velocity distribution as given by (10) the calculation of the boundary layer flow will be carried out by successive approximations. That is, we decompose the velocity components u, w and v into sums

$$\begin{aligned} u &= u_0 + u_1 + u_2 + \cdots \\ w &= w_0 + w_1 + w_2 + \cdots \\ v &= v_0 + v_1 + v_2 + \cdots \end{aligned} \tag{11}$$

where u_0, w_0, v_0 are the solutions of first-order approximation, u_1, w_1, v_1 are the solutions of second-order approximation and so on, and that

$$\begin{aligned} u_0 &\gg u_1 \gg u_2 \gg \cdots \\ w_0 &\gg w_1 \gg w_2 \gg \cdots \\ v_0 &\gg v_1 \gg v_2 \gg \cdots \end{aligned} \tag{12}$$

Following Schlichting we state that it is possible to solve equations (4) to (6) using the method of successive approximation of reference (1) for the case of periodic boundary layers also if

$$\left| u_e \frac{\partial u_e}{\partial s} \right| \ll \left| \frac{\partial u_e}{\partial t} \right|$$

which leads to the condition that $S/d \ll 1$ where S is the amplitude of oscillations and d is the linear dimension of the body. That is, the proposed method of solution may be used in cases where the amplitude of oscillations is small compared to the dimensions of the body. Hence using (11) in (4)–(6) we obtain the equations for the first-order approximation as [reference (1)]

$$\frac{\partial u_0}{\partial t} - \nu \frac{\partial^2 u_0}{\partial n^2} = \frac{\partial u_e}{\partial t}, \tag{13}$$

$$\frac{\partial (h_2 u_0)}{\partial s} + \frac{\partial (h_2 v_0)}{\partial n} = 0, \tag{14}$$

$$\frac{\partial w_0}{\partial t} - \nu \frac{\partial^2 w_0}{\partial n^2} = \frac{u_0{}^2 - u_e{}^2}{h_1} \frac{\partial h_1}{\partial z}, \tag{15}$$

with the boundary conditions

$$u_0 = v_0 = w_0 = 0 \quad \text{at } n = 0$$

$$u_0 \to u_e \text{ as } n \to \infty. \tag{16}$$

The equations for the second-order approximation are

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial n^2} = u_e \frac{\partial u_e}{\partial s} - u_0 \frac{\partial u_0}{\partial s} - v_0 \frac{\partial u_0}{\partial n}, \tag{17}$$

$$\frac{\partial (h_2 u_1)}{\partial s} + \frac{\partial (h_2 v_1)}{\partial n} = 0 \tag{18}$$

$$\frac{\partial w_1}{\partial t} - \nu \frac{\partial^2 w_1}{\partial n^2} = \frac{(2u_0 u_1 + u_1{}^2)}{h_1} \frac{\partial h_1}{\partial z} - (u_0 + u_1) \frac{\partial w_0}{\partial s} - (v_0 + v_1) \frac{\partial w_0}{\partial n}, \tag{19}$$

and the corresponding boundary conditions are

$$u_1 = v_1 = w_1 = 0 \quad \text{at } n = 0 \text{ and } \infty. \tag{20}$$

SOLUTIONS

First approximation.—Introducing a dimensionless co-ordinate y defined by

$$y = n \sqrt{\frac{\omega}{\nu}} \tag{21}$$

and assuming that the first-order velocity u_o is of the form

$$u_0 = U_0 \phi_0'(y) e^{i\omega t} \tag{22}$$

we get from (14)

$$v_0 = -\sqrt{\frac{\nu}{\omega}} \frac{1}{h_2} \frac{\partial (h_2 U_0)}{\partial s} \phi_0(y) e^{i\omega t} \tag{23}$$

where

$$\phi_0'(0) = 0, \ \phi_0(0) = 0, \ \phi_0'(\infty) = 1. \tag{24}$$

Substituting (10), (21) and (22) in (13) we obtain the following differential equation for ϕ_0',

$$\phi_0''' - i\phi_0' = -i \tag{25}$$

with the boundary conditions (24) equation (25) has the solution

$$\phi_0' = 1 - e^{-x} \tag{26}$$

where

$$x = ay \quad \text{and} \quad a = \frac{1+i}{\sqrt{2}}. \tag{27}$$

Hence we obtain (in real notation)

$$u_0(s, z, y, t) = U_0 \left[\cos \omega t - e^{-y/\sqrt{2}} \cos \left(\omega t - \frac{y}{\sqrt{2}}\right)\right], \tag{28}$$

and integrating (26) once again and using (24) we get

$$\phi_0 = \frac{x - 1 + e^{-x}}{a}, \tag{29}$$

which, on substitution in (23), gives

$$v_0(s, z, y, t) = -\sqrt{\frac{\nu}{\omega}} \frac{1}{h_2} \frac{\partial (h_2 U_0)}{\partial s} \left[y \cos \omega t - \cos \left(\omega t - \frac{\pi}{4}\right) + e^{-y/\sqrt{2}} \cos \left(\omega t - \frac{\pi}{4} - \frac{y}{\sqrt{2}}\right)\right]. \tag{30}$$

First-order cross-flow.—The differential equation governing the first-order cross-flow w_0 is the equation (15), *viz.*,

$$\frac{\partial w_0}{\partial t} - \nu \frac{\partial^2 w_0}{\partial n^2} = \frac{u_0^2 - u_e^2}{h_1} \frac{\partial h_1}{\partial z}, \tag{14}$$

with the boundary conditions

$$w_0 = 0 \quad \text{at} \quad n = 0, \infty. \tag{15}$$

Before assuming some form for w_0 it must be observed that the right-hand side of (14) contributes terms like $\cos^2 \omega t$, $\sin^2 \omega t$ which in turn can be reduced to terms with $\cos 2\omega t$, $\sin 2\omega t$ and steady state, *i.e.*, time-independent terms. Hence, under these circumstances we can express the first-order cross-flow as

$$w_0 (s, z, y, t) = \frac{U_0^2}{2\omega h_1} \frac{\partial h_1}{\partial z} [\psi_{0a} (y)\, e^{2i\omega t} + \psi_{0b} (y)], \tag{31}$$

where $\psi_{0a} (y)$ and $\psi_{0b} (y)$ denote respectively the periodic and the steady state contributions. Substituting (31) in (15) we get

$$\psi_{0a}'' - 2i\psi_{0a} = 1 - \phi_0'^2 \tag{32}$$

$$\psi_{0b}'' = 1 - \phi_0' \bar{\phi}_0 \tag{33}$$

where the bar indicates the complex conjugate. The actual boundary conditions are that the cross-flow should vanish both at the wall and at large distances from it. But as will be seen that though the fluctuating part can be made equal to zero at both the points ($y = 0, \infty$) the steady part can be made zero only at the wall and not at infinity. At the most we can see that it is finite at infinity.* Hence, with this point in view we obtain the solutions for (32) and (33) as

$$\psi_{0a} = i\,[2e^{-x} + 0\cdot 5\, e^{-2x} - 2\cdot 5\, e^{-\sqrt{2}\, x}) \tag{34}$$

$$\psi_{0b} = 0\cdot 5\,(1 - e^{-\sqrt{2}\, y}) - 2e^{-y/\sqrt{2}} \sin \frac{y}{\sqrt{2}} \tag{35}$$

so that

$$\psi_{0b} (\infty) = 0\cdot 5.$$

Hence, the cross-flow is seen to contain a steady state term which does not vanish outside the boundary layer. Now the actual expression for this first-order cross-flow is given by

$$w_0(s, z, y, t)$$
$$= \frac{U_0^2}{2\omega h_1}\frac{\partial h_1}{\partial z}\left[2e^{-y/\sqrt{2}}\sin\left(\frac{y}{\sqrt{2}} - 2\omega t\right)\right.$$
$$+ 0\cdot 5\, e^{-\sqrt{2}y}\sin(\sqrt{2}\,y - 2\omega t) - 2\cdot 5\, e^{-y}\sin(y - 2\omega t)$$
$$\left. + 0\cdot 5\,(1 - e^{-\sqrt{2}y}) - 2e^{-y/\sqrt{2}}\sin\frac{y}{\sqrt{2}}\right]. \quad (36)$$

The streamwise and the crosswise velocity profiles

$$u_0{}^* = \frac{u_0}{U_0} \quad \text{and} \quad \omega_0{}^* = \frac{w_0}{\dfrac{U_0^2}{2\omega h_1}\dfrac{\partial h_1}{\partial z}}$$

given by equations (28) and (36) respectively are plotted in Figs. 1, 2*a* and 2*b*. In Fig. 2*a* the fluctuating velocity component $w_{0a}{}^*$ of the

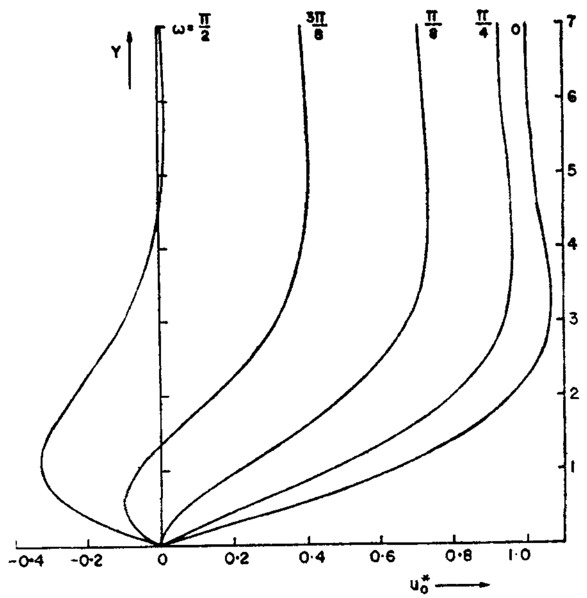

FIG. 1. First order streamwise velocity profiles.
In curves 1, 3 and 4 for $\omega = \pi/2, \pi/8, \pi/4$ read respectively $\omega t = \pi/2, \pi/4$ and $\pi/8$.

*As has been done by Stuart,[8] it may be necessary to make a separate analysis assuming an outer boundary layer in between the potential flow and the present boundary layer, in which the flow is 'streaming' steady and satisfying the boundary condition of zero velocity at infinity.

cross-flow for different ωt and its steady velocity component $w_{0b}{}^*$ are plotted separately whereas in Fig. 2*b* the resultant cross-flow for the same values of ωt are plotted. In both these figures the dotted line indicates the steady component. It may be recalled that in reference (2) (and also here) this interesting feature of the oscillatory potential flow, namely, the existence of a steady stream outside the boundary layer is observed only in the second approximation for the velocity and there an experimental evidence is also provided. The existence of the steady stream can be explained as due to the fact that the negligence of inertial terms in the first-order equations is valid only very near to the wall and the Reynolds stresses in this oscillatory boundary layer cause the formation of steady streaming flow: and there is an outer boundary layer within which this steady streaming velocity decays to zero. The thickness of this outer layer is large compared with that of the inner oscillatory layer, but small compared with a typical dimension of the body. These observations are also confirmed by Riley.[4] But an evidence for steady 'streaming motion' in the cross-flow direction is yet to forthcome.

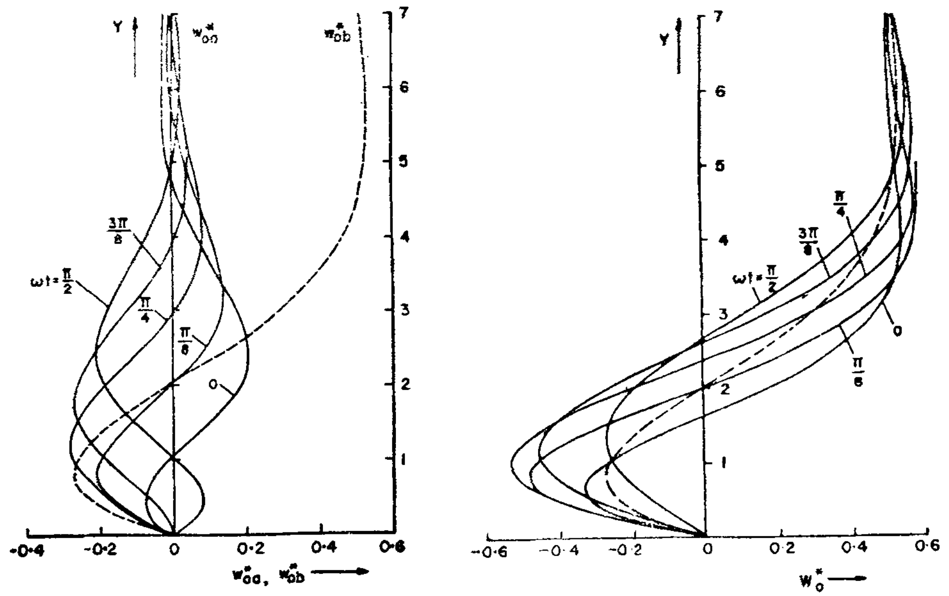

FIG. 2 (*a*)　　　　FIG. 2 (*b*)

FIG. 2 (*a*). First order crosswise velocity profiles.
($w_{0a}{}^*$ = fluctuating part, $w_{0b}{}^*$ = steady part)

FIG. 2 (*b*). First order crosswise velocity profiles (Resultant).

Second approximation.—The differential equations governing the second approximation are (17), (18) and (19). Using real parts of (10), (22) and (23) to calculate the right-hand side of (17) we obtain the differential equation for u_1 as

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial n^2}$$

$$= \frac{1}{2}\Big[U_0 \frac{\partial U_0}{\partial s}\Big\{(1 - \phi_0'^2 + \phi_0\phi_0'') e^{2i\omega t}$$

$$+ \Big(1 - \phi_0'\bar{\phi}_0' + \frac{\phi_0\bar{\phi}_0'' + \bar{\phi}_0\phi_0''}{2}\Big)\Big\} + \frac{U_0^2}{h_2}\frac{\partial h_2}{\partial s}$$

$$\times \Big\{\phi_0\phi_0'' e^{2i\omega t} + \frac{\phi_0\bar{\phi}_0'' + \bar{\phi}_0\phi_0''}{2}\Big\}\Big]. \tag{37}$$

Hence, it would be appropriate to write the solution of u_1 in the form

$$u_1 = \frac{1}{2\omega}\Big[e^{2i\omega t}\Big\{U_0 \frac{\partial U_0}{\partial s}\phi_{11}'(y) + \frac{U_0^2}{h_2}\frac{\partial h_2}{\partial s}\phi_{12}'(y)\Big\}$$

$$+ \Big\{U_0 \frac{\partial U_0}{\partial s}\phi_{13}'(y) + \frac{U_0^2}{h_2}\frac{\partial h_2}{\partial s}\phi_{14}'(y)\Big\}\Big], \tag{38}$$

with this we get the following four differential equations:

$$\phi_{11}''' - 2i\phi_{11}' = -(1 - \phi_0'^2 + \phi_0\phi_0''), \tag{39}$$

$$\phi_{12}''' - 2i\phi_{12}' = -\phi_0\phi_0'', \tag{40}$$

$$\phi_{13}''' = -\Big(1 - \phi_0'\bar{\phi}_0' + \frac{\phi_0\bar{\phi}_0'' + \bar{\phi}_0\phi_0''}{2}\Big), \tag{41}$$

$$\phi_{14}''' = -\frac{\phi_0\bar{\phi}_0'' + \bar{\phi}_0\phi_0''}{2}. \tag{42}$$

The boundary conditions are:

$$\phi_{11}(0) = \phi_{11}'(0) = \phi_{11}'(\infty) = 0$$

$$\phi_{12}(0) = \phi_{12}'(0) = \phi_{12}'(\infty) = 0$$

$$\phi_{13}(0) = \phi_{13}'(0) = \phi_{14}(0) = \phi_{14}'(0) = 0$$

$$\phi_{13}(\infty),\ \phi_{14}(\infty) < \infty, \tag{43}$$

Equations (39)–(43) have solutions:

$$\phi_{11}' = i\,[(1-x)\,e^{-x} - e^{-\sqrt{2}x})\tag{44}$$

$$\phi_{12}' = i\left[(3-x)\,e^{-x} + \frac{e^{-2x}}{2} - \frac{7}{2}\,e^{-\sqrt{2}x}\right]\tag{45}$$

$$\phi_{13}' = -\frac{3}{2} + \frac{e^{-\sqrt{2}y}}{2} + e^{-y/\sqrt{2}}\left[4\sin\frac{y}{\sqrt{2}} + \cos\frac{y}{\sqrt{2}} - \frac{y}{\sqrt{2}} \times \left(\cos\frac{y}{\sqrt{2}} - \sin\frac{y}{\sqrt{2}}\right)\right]\tag{46}$$

$$\phi_{14}' = -1 + e^{-y/\sqrt{2}}\left[\cos\frac{y}{\sqrt{2}} + 2\sin\frac{y}{\sqrt{2}} - \frac{y}{\sqrt{2}} \times \left(\cos\frac{y}{\sqrt{2}} - \sin\frac{y}{\sqrt{2}}\right)\right].\tag{47}$$

From the above four solutions the functions ϕ_{11}, ϕ_{12}, ϕ_{13} and ϕ_{14} can be obtained by integrating once. Then equation (18) gives the solution for v_1 in the form

$$v_1 = -\sqrt{\frac{\nu}{\omega}}\,\frac{1}{2\omega h_2}\left[\frac{\partial\left(h_2 U_0 \frac{\partial U_0}{\partial s}\right)}{\partial s}(\phi_{11}\,e^{2i\omega t} + \phi_{13}) + \frac{\partial\left(U_0^{2}\frac{\partial h_2}{\partial s}\right)}{\partial s} \times (\phi_{12}\,e^{2i\omega t} + \phi_{14})\right].\tag{48}$$

It may be observed in the right-hand side of the equation (37) that the curvature $\partial h_2/\partial s$ brings in additional functions ϕ_{12} and ϕ_{14} into existence. These are absent in a two-dimensional rectangular or cylindrical system of co-ordinates. This is an additional feature of the second-order stream-wise velocity component in general three-dimensional flow.

Second-order cross-flow.—The differential equation for the second-order cross-flow is given by the equation (19). From the already known solutions of u_0, v_0, w_0, u_1 and v_1 the right-hand side of this equation can be easily written down in terms of ϕs and ψs as:

$$= \sum_{j=1}^{7} F_j [\chi_j + \chi_{(j+11)} e^{2i\omega t} + \chi_{(j+22)} e^{4i\omega t}]$$

$$+ \sum_{j=8}^{11} F_j [\chi_j e^{i\omega t} + \chi_{(j+11)} e^{3i\omega t}] \tag{49}$$

where

$$F_1 = \frac{U_0^2}{4\omega^2 h_1} \frac{\partial h_1}{\partial z} \left(\frac{\partial U_0}{\partial s}\right)^2, \qquad F_2 = \frac{U_0^4}{4 h_1 h_2^2 \omega^2} \frac{\partial h_1}{\partial z} \left(\frac{\partial h_2}{\partial s}\right)^2,$$

$$F_3 = \frac{2U_0^3}{4\omega^2 h_1 h_2} \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial s} \frac{\partial U_0}{\partial s}, \qquad F_4 = -\frac{U_0}{4\omega^2} \frac{\partial U_0}{\partial s} \frac{\partial \left(\frac{U_0^2}{h_1} \frac{\partial h_1}{\partial z}\right)}{\partial s},$$

$$F_5 = -\frac{U_0^2}{4\omega^2 h_2} \frac{\partial h_2}{\partial s} \frac{\partial \left(\frac{U_0^2}{h_1} \frac{\partial h_1}{\partial z}\right)}{\partial s}, \quad F_6 = \frac{U_0^2}{4\omega^2 h_1 h_2} \frac{\partial h_1}{\partial z} \frac{\partial \left(U_0 h_2 \frac{\partial U_0}{\partial s}\right)}{\partial s},$$

$$F_7 = \frac{U_0^2}{4\omega^2 h_1 h_2} \frac{\partial h_1}{\partial z} \frac{\partial \left(U_0^2 \frac{\partial h_2}{\partial s}\right)}{\partial s}, \quad F_8 = \frac{U_0^2}{\omega h_1} \frac{\partial h_1}{\partial z} \frac{\partial U_0}{\partial s},$$

$$F_9 = \frac{U_0^3}{\omega h_1 h_2} \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial s}, \qquad F_{10} = -\frac{U_0}{2\omega} \frac{\partial \left(\frac{U_0^2}{h_1} \frac{\partial h_1}{\partial z}\right)}{\partial s},$$

$$F_{11} = \frac{U_0^2}{2\omega h_1 h_2} \frac{\partial h_1}{\partial z} \frac{\partial (h_2 U_0)}{\partial s},$$

and

$$\chi_1 = \phi_{13}'^2 + \frac{\phi_{11}'\bar{\phi}_{11}'}{2}, \qquad \chi_2 = \frac{\phi_{12}'\bar{\phi}_{12}'}{2} + \phi_{14}'^2,$$

$$\chi_3 = \phi_{13}'\phi_{14}' + \frac{\phi_{11}'\bar{\phi}_{12}' + \bar{\phi}_{11}'\phi_{12}'}{4},$$

$$\chi_4 = \psi_{0b}\phi_{13}' + \frac{\psi_{0a}\bar{\phi}_{11}' + \bar{\psi}_{0a}\phi_{11}'}{4},$$

$$\chi_5 = \psi_{0b}\phi_{14}' + \frac{\psi_{0a}\bar{\phi}_{12}' + \bar{\psi}_{0a}\phi_{12}'}{4},$$

$$\chi_6 = \psi_{0b}'\phi_{13} + \frac{\phi_{11}\bar{\psi}_{0a}' + \bar{\phi}_{11}\psi_{0a}'}{4},$$

$$\chi_7 = \psi_{0b}'\phi_{14} + \frac{\phi_{12}\bar{\psi}_{0a}' + \bar{\phi}_{12}\psi_{0a}'}{4}, \quad \chi_8 = \phi_{13}'\phi_0' + \frac{\bar{\phi}_0'\phi_{11}'}{2},$$

$$\chi_9 = \phi_{14}'\phi_0' + \frac{\bar{\phi}_0'\phi_{12}'}{2}, \qquad \chi_{10} = \psi_{0b}\phi_0' + \frac{\bar{\phi}_0'\psi_{0a}}{2},$$

$$\chi_{11} = \psi_{0b}'\phi_0 + \frac{\bar{\phi}_0\psi_{0a}'}{2}, \quad \chi_{12} = 2\phi_{13}'\phi_{11}', \quad \chi_{13} = 2\phi_{14}'\phi_{12}',$$

$$\chi_{14} = \phi_{13}'\phi_{12}' + \phi_{14}'\phi_{11}', \quad \chi_{15} = \psi_{0b}\phi_{11}' + \phi_{13}'\psi_{0a},$$

$$\chi_{16} = \psi_{0b}\phi_{12}' + \phi_{14}'\psi_{0a}, \quad \chi_{17} = \phi_{13}\psi_{0a}' + \phi_{11}\psi_{0b}',$$

$$\chi_{18} = \phi_{14}\psi_{0a}' + \phi_{12}\psi_{0b}', \quad \chi_{19} = \frac{\phi_0'\phi_{11}'}{2}, \quad \chi_{20} = \frac{\phi_0'\phi_{12}'}{2},$$

$$\chi_{21} = \frac{\phi_0'\psi_{0a}}{2}, \quad \chi_{22} = \frac{\phi_0\psi_{0a}'}{2}, \quad \chi_{23} = \frac{\phi_{11}'^2}{2}, \quad \chi_{24} = \frac{\phi_{12}'^2}{2},$$

$$\chi_{25} = \frac{\phi_{11}'\phi_{12}'}{2}, \quad \chi_{26} = \frac{\psi_{0a}\phi_{11}'}{2}, \quad \chi_{27} = \frac{\psi_{0a}\phi_{12}'}{2}, \quad \chi_{28} = \frac{\phi_{11}\psi_{0a}'}{2},$$

$$\chi_{29} = \frac{\phi_{12}\psi_{0a}'}{2}. \tag{50}$$

Hence, it is seen that the solution of the second-order cross-flow has to be written in the form

$$w_1 = \sum_{j=1}^{7} F_j\psi_j + e^{i\omega t}\sum_{j=8}^{11} F_j\psi_j + e^{2i\omega t}\sum_{j=1}^{7} F_j\psi_{(j+11)}$$
$$+ e^{3i\omega t}\sum_{j=8}^{11} F_j\psi_{(j+11)} + e^{4i\omega t}\sum_{j=1}^{7} F_j\psi_{(j+22)}. \tag{51}$$

The second-order cross-flow is thus found to consist of fluctuating flow with frequencies ω, 2ω, 3ω and 4ω in addition to a steady streaming component which does not vanish outside the boundary layer. On substitution of (51) in the equation (19) we obtain twenty-nine ordinary linear second-order differential equations with constant coefficients which are easily solvable analytically. They can be formally written as:

$$\psi_j'' = -\chi_j$$

$$\psi_k'' - i\psi_k = - \chi_k$$
$$\psi''_{(j+11)} - 2i\psi_{(j+11)} = - \chi_{(j+11)}$$
$$\psi''_{(k+11)} - 3i\psi_{(k+11)} = - \chi_{(k+11)}$$
$$\psi''_{(j+22)} - 4i\psi_{(j+22)} = - \chi_{(j+22)} \qquad (52)$$

where $j = 1, 2, 3, \cdots 7$; $k = 8, 9, 10$ and 11, and the χ_s are as given in equation (50).

REFERENCES

1. Eichelbrenner, E. A. and Askovic, R. .. "Sur une methode approchee de traiter les couches limites laminaires en regime instationnaire dun ecoulement dans un fluide incompressible a trois dimensions," *J. de Mecanique*, 1967, **6**, 353.

2. Schlichting, H. .. *Boundary Layer Theory. Chapter XI.* McGraw-Hill, 1960.

3. Stuart, J. T. .. "Double boundary layers in oscillatory viscous flow," *N.P.L. Aero. Report*, 1965, p. 1153.

4. Riley, N. .. "Oscillating viscous flows," *Mathematika*, 1965, **12**, 161.

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