SLOWLY OSCILLATING FUNCTIONS AND A GENERALIZATION OF QUASI-MONOTONE COEFFICIENTS

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ABSTRACT

We first state a conjecture due to Yong concerning equality of two classes of functions, and then give examples to disprove the conjecture. Later we extend some theorems concerning integrability of Fourier series.

SECTION 1

In this paper we extend several theorems of Yong[3] and also prove a conjecture of his. One of the main tools is the class of slowly oscillating functions. A continuous, positive-valued function s, defined for all large values, is called slowly oscillating if it satisfies the condition $\lim_{x\to\infty} s(kx)/s(x) = 1$ for every fixed $k > 0$. Some of the properties possessed by slowly oscillating functions are set forth below.

(i) s is slowly oscillating if and only if

$$s(x) = \mathrm{F}(x) \exp.\left[\int_a^x \frac{\delta(t)}{t}\, dt\right],$$

where F is a positive-valued continuous function which tends to a positive, finite limit, a is some positive constant, and δ is a continuous function which tends to zero.

(ii) $x^{-a} s(x) \to 0$ and $x^{a} s(x) \to \infty$ for every fixed $a > 0$.

(iii) if $a > 0$, then there are positive, finite numbers A_1 and A_2 such that for every natural number n

$$\mathrm{A}_1 n^a s(n) \leq \sum_1^n k^{a-1} s(k) \leq \mathrm{A}_2 n^a s(n).$$

(iv) if $\alpha > 0$, then as $q \to \infty$

$$\min_{q \leq x < \infty} [x^{\alpha} s(x)] \sim q^{\alpha} s(q).$$

(v) the sum and the product (but not necessarily the difference) of any two slowly oscillating functions is itself slowly oscillating.

Section 2

The first theorem in Yong's paper is concerned with two classes of functions, and at the end of the proof of his Theorem 1 he states that he has been unable to decide which inclusion relations hold between the two classes. He conjectures, however, that the classes are not comparable. We prove this conjecture by exhibiting two functions, each of which is in one class but not the other. For both classes the functions must be defined for all large values, be positive, and satisfy

$$\sum_{K}^{\infty} \frac{1}{[nf(n)]} < \infty. \tag{1}$$

(Here and throughout this paper K denotes some unspecified positive constant whose value may change from one occurrence to the next.) The two classes are then defined respectively as those functions f which satisfy the above conditions and are

(A) slowly oscillating,

(B) non-decreasing for all large values and satisfy

$$f(x^k) \leq cf(x) \tag{2}$$

for some fixed $k > 1$, $c = c(k)$ some positive constant. We show first that (A) is not contained in (B). Letting

$$f(x) = \exp. \left[\int_{10}^{x} \frac{dt}{\{t\mathrm{L}(t)\}} \right]$$

[in this section $\mathrm{L}(x) = \log\log x$] we know that f is slowly oscillating by (i). Now for all large x we have

$$f(x) \geq \exp. \left[\left(\frac{1}{\mathrm{L}(x)}\right) \int_{10}^{x} \frac{dt}{t} \right]$$

$$= \left(\frac{x}{10}\right)^{1/\mathrm{L}(x)} \geq \log^2 x,$$

so (1) holds and f is in A. Now let $k > 1$. Then

$$\frac{f(x^k)}{f(x)} = \exp.\left[\int_x^{x^k} \frac{dt}{\{t\mathrm{L}(t)\}}\right]$$

$$\geq \exp.\left[\left(\frac{1}{\mathrm{L}(x^k)}\right)\int_x^{x^k} \frac{dt}{t}\right] = \exp.\left[\frac{(k-1)\log x}{\log k + \mathrm{L}(x)}\right],$$

which tends to infinity with x. Thus (2) does not hold and f is not in (B).

We turn now to an example of a function which is in (B) but not (A). Let $\mathrm{E}(x) = e^{e^x}$ and define I_n to be the closed interval $[\mathrm{E}(n/2), 1 + \mathrm{E}(n/2)]$. Define the function h for $t \geq 10$ by $h(t) = \mathrm{E}(n/2)$ if $t \in \mathrm{I}_n$ and $h(t) = 0$ otherwise. Finally, define g by

$$g(x) = \exp.\left[\int_{10}^{x} \frac{h(t)}{t}\, dt\right].$$

We proceed to show that g is the desired function. We note first that for every positive integer n we have

$$\frac{2}{3} < \int_{\mathrm{I}_n} \frac{h(t)}{t}\, dt < 1. \tag{3}$$

If g were in (A) it would have to satisfy $g(kx)/g(x) \to 1$ for every fixed $k > 0$, that is $\int_x^{kx} h(t)/t\, dt \to 0$. But choosing $k > 1$ and letting $x \to \infty$ through the values $\mathrm{E}(n/2)$ yields by (3)

$$\limsup_{x \to \infty} \int_x^{kx} \frac{h(t)}{t}\, dt \geq \frac{2}{3}.$$

Thus g is not slowly oscillating, hence not in (A). To show that g is in (B) we observe that $h(t) \geq 0$, so g is non-decreasing and positive. Now $h(t) > 0$ in $[x, x^k]$ if and only if $x - 1 \leq \mathrm{E}(n/2) \leq x^k$, that is $\mathrm{L}(x-1) \leq n/2 \leq \log k + \mathrm{L}(x)$. But $\log k + \mathrm{L}(x) - \mathrm{L}(x-1)$ is bounded, so

$h(t) > 0$ in $[x, x^k]$ at most, say T times (T being independent of x). We thus may say, by (3), that

$$\frac{g(x^k)}{g(x)} = \exp.\left[\int_x^{x^k} \frac{h(t)}{t}\,dt\right] \leq \exp.[T],$$

and so (2) holds. It remains only to show that (1) holds also. Now $9 \leq E(n/2) \leq x$ if and only if $2L(9) \leq n \leq 2L(x)$, so $h(t) > 0$ in $[10, x]$ at least $[2L(x) - 2L(9) - 3]$ times (here the brackets denote the greatest integer function). But for all large x this last expression is larger than $(7/4)\,L(x)$, so for such values of x we have by (3)

$$g(x) \geq \exp.\left[\left(\frac{2}{3}\right)\left(\frac{7}{4}\right)L(x)\right] = (\log x)^{7/6}.$$

Thus (1) holds, g is in (B) and Yong's conjecture is proved.

Section 3

We now turn to extension of Yong's results by extending the concept of quasi-monotone sequences. A sequence $\{a_k\}$ of positive terms is called quasi-monotone if $a_{k+1} \leq a_k(1 + \alpha/k)$ for some $\alpha \geq 0$. We consider sequences which satisfy

$$a_{k+1} \leq a_k\left(1 + \frac{S(k)}{k}\right), \tag{4}$$

where S is a slowly oscillating function which increases to infinity. By (ii) we have $S(k)/k \to 0$.

Theorem 1.—Let $p > 0$, L be a slowly oscillating function, $\{a_k\}$ and S as above. Then

(*a*) if

$$\sum_1^\infty k^{p-1} L(k) S^2(k)\, a_k < \infty$$

then

$$\sum_1^\infty k^p L(k) S(k) \,|\, a_k - a_{k+1} \,| < \infty,$$

(*b*) if

$$\sum_1^\infty k^p L(k) S(k) \,|\, a_k - a_{k+1} \,| < \infty$$

then

$$\sum_1^\infty k^{p-1} \mathrm{L}(k) \mathrm{S}(k) a_k < \infty.$$

Proof.—(*a*) By partial summation

$$\sum_1^q n^{p-1} \mathrm{L}(n) \mathrm{S}(n) a_n = \sum_1^{q-1} s_n (a_n - a_{n+1}) + s_q a_q, \tag{5}$$

where

$$s_n = \sum_1^n k^{p-1} \mathrm{L}(k) \mathrm{S}(k).$$

We define

$$u_n = n + \left[\frac{n}{\mathrm{S}(n)}\right],$$

where $[x]$ is the greatest integer not exceeding x. Now consider

$$t_n = \sum_n^{u_n} k^{p-1} \mathrm{L}(k) \mathrm{S}^2(k) a_k.$$

we have, for each k satisfying $n \le k \le u_n$, that $k/u_n \ge n/u_n \to 1$, so

$$\left(\frac{k}{u_n}\right)^{p-1} \ge \mathrm{K} > 0. \tag{6}$$

Using the representation in (i) we may write

$$\frac{\mathrm{L}(k)}{\mathrm{L}(u_n)} = \frac{\mathrm{F}(k)}{\mathrm{F}(u_n)} \exp.\left[\int_k^{u_n} \frac{-\delta(t)}{t}\, dt\right].$$

But for all large n, $\mathrm{F}(k)/\mathrm{F}(u_n) \ge \frac{1}{2}$ and $-\delta(t) > -1$, so

$$\frac{\mathrm{L}(k)}{\mathrm{L}(u_n)} \ge k \exp.\left[-\log \frac{u_n}{k}\right] = \mathrm{K}\, \frac{k}{u_n} \to \mathrm{K}.$$

Thus by (v)

$$\frac{\mathrm{L}(k)\, \mathrm{S}^2(k)}{\mathrm{L}(u_n)\, \mathrm{S}^2(u_n)} \ge \mathrm{K} > 0. \tag{7}$$

For the last part of t_n, we have

$$a_k \geq \frac{a_{k+1}}{1+\frac{S(k)}{k}} \geq \dots \geq \frac{a(u_n)}{\left(1+\frac{S(k)}{k}\right)\dots\left(1+\frac{S(u_n)}{u_n}\right)}.$$

By (iv), for all large n each factor of the last denominator is dominated by

$$1+\left\{\max_{n\leq k<\infty} \frac{S(k)}{k}\right\} \leq 1+\frac{2S(n)}{n},$$

and

$$\log\left\{1+\frac{2S(n)}{n}\right\}^{1+[n/S(n)]} \leq \left\{1+\left[\frac{n}{S(n)}\right]\right\}\frac{2S(n)}{n} \leq K,$$

so

$$a_k \geq K a(u_n). \tag{8}$$

Thus from (6), (7), and (8), we have

$$\begin{aligned} t_n &\geq \frac{K u_n^{p-1} L(u_n) S^2(u_n) a(u_n) n}{S(n)} \\ &\geq K u_n^p L(u_n) S(u_n) a(u_n). \end{aligned}$$

However, by hypothesis $t_n \to 0$, so the sequence $\{n^p L(n) S(n) a_n\}$ tends to zero as n tends to infinity through the sequence $\{u_n\}$. But since S increases we have

$$\begin{aligned} u_{n+1} - u_n &\leq 1 + \left[\frac{n+1}{S(n+1)} - \frac{n}{S(n)}\right] \\ &\leq 1 + \frac{1}{S(n+1)} + \frac{n[S(n)-S(n+1)]}{S(n)S(n+1)} \\ &\leq 1 + \frac{1}{S(n+1)} \to 1, \end{aligned} \tag{9}$$

so the gaps in $\{u_n\}$ are bounded and $n L(n) S(n) a_n \to 0$. Now by (iii)

$$s_q a_q \leq A_2 q^p L(q) S(q) a_q \to 0,$$

and so from (5)

$$\sum_1^\infty n^{p-1} L(n) S(n) a_n = \sum_1^\infty s_n (a_n - a_{n+1}). \tag{10}$$

With Yong[3] and Shah[2] for the sequence $\{a_k\}$ we define two new sequences $\{n_j\}$ and $\{p_j\}$ by saying that $a_{k+1} \geq a_k$ if $n_j \leq k \leq n_j + p_j - 1$ and $a_{k+1} < a_k$ if $n_j + p_j \leq k \leq n_{j+1} - 1$. Writing the right side of (10) as two sums and transposing one of them gives

$$\sum_1^\infty s_n (a_n - a_{n+1}) + \sum_{j=1}^\infty \sum_{k=n_j}^{n_j+p_j-1} s_k (a_{k+1} - a_k)$$

$$= \sum_{j=1}^\infty \sum_{k=n_j+p_j}^{n_{j+1}-1} s_k (a_k - a_{k+1}). \tag{11}$$

This is permissible since by (4) and (iii) the second sum in (11) is dominated by

$$\sum_1^\infty \sum_{n_j}^{n_j+p_j-1} A_2 k^p L(k) S(k) \frac{a_k S(k)}{k} \leq A_2 \sum_1^\infty k^{p-1} L(k) S^2(k) a_k$$

which is finite by hypothesis. The first sum in (11) is finite by (10), hence so is the third sum. However, from (iii),

$$s_k \mid a_k - a_{k+1} \mid \geq A_1 k^p L(k) S(k) \mid a_k - a_{k+1} \mid,$$

so

$$\sum_1^\infty k^p L(k) S(k) \mid a_k - a_{k+1} \mid < \infty,$$

which is the conclusion of (*a*).

To prove (*b*) we again appeal to partial summation and (iii) and obtain

$$\sum_1^q n^{p-1} S(n) L(n) a_n$$

$$= \sum_1^{q-1} s_n (a_n - a_{n+1}) + s_q a_q \leq A_2 \sum_1^{q-1} n^p L(n) S(n) \mid a_n - a_{n+1} \mid$$
$$+ s_q a_q. \tag{12}$$

But from (iv)

$$\sum_q^\infty n^p L(n) S(n) \mid a_n - a_{n+1} \mid$$

$$\geq \sum_q^\infty \min_{q \leq n < \infty} \{n^p L(n) S(n)\} \mid a_n - a_{n+1} \mid$$

$$\geq Kq^p L(q) S(q) \sum_q^\infty | a_n - a_{n+1} |$$

$$\geq Kq^p L(q) S(q) a_q,$$

so from the hypothesis $n^q L(n) S(n) a_n \to 0$. Thus $s_q a_q$ is bounded and the conclusion of (*b*) follows.

It is natural to ask if $S^2(k)$ may be replaced by $S(k)$ in the hypothesis of part (*a*) of the theorem. We give here an example to show that it may not. Let $L(k) = \log k$, $S(k) = (\log k)^{2\epsilon}$ for some fixed $\epsilon > 0$, $p = 1$, and define $\{a_k\}$ by $a_1 = a_2 = 1$ and $k \geq 3$

$$a_k = \frac{1}{k(\log k)^{2+3\epsilon}},$$

if k is odd

$$a_k = \frac{1}{k(\log k)^{2+3\epsilon}}\left[1 + \frac{(\log k)^{2\epsilon}}{k}\right],$$

if k is even. Then $\{a_k\}$ satisfies (4), L and S are slowly oscillating, and $\Sigma k^{p-1} L(k) S(k) a_k < \infty$. However for k even

$$| a_k - a_{k+1} | \geq \frac{1}{k(\log k)^{2+3\epsilon}} \frac{(\log k)^{2\epsilon}}{k} = \frac{1}{k^2 (\log k)^{2+\epsilon}},$$

and so

$$\sum_1^N k^p L(k) S(k) | a_k - a_{k+1} | \geq \sum_{\substack{1 \\ k \text{ even}}}^N \frac{1}{k(\log k)^{1-\epsilon}},$$

which tends to infinity with N.

We note that, by the theorem, for such an example as this we must have $\Sigma k^{p-1} L(k) S^2(k) a_k = \infty$. That this last relation holds follows from

$$\sum_3^\infty \frac{(\log k)(\log k)^{4\epsilon}}{k(\log k)^{2+3\epsilon}} = \infty.$$

The only place where the hypothesis that S increases is used is in showing that

$$\frac{n+1}{S(n+1)} - \frac{n}{S(n)}$$

is bounded [*see* (9)]. That we may not obtain this result by assuming only that $S(x) \to \infty$ instead of $S(x) \uparrow \infty$ follows from the example given here. Let $S(x) = \log x + \cos \pi x$, which is slowly oscillating. Then for n even we have

$$\frac{n+1}{S(n+1)} - \frac{n}{S(n)}$$

$$= \frac{2n + \log n + 1 + n[\log n - \log(n+1)]}{[\log(n+1) - 1][\log n + 1]} \to \infty.$$

We may use Theorem 1 to obtain integrability theorems similar to those of Yong and Boas [*see* (3) and (1)]. These results are stated here but not proved.

Theorem 2.—Let $\{a_k\}$ satisfy (4) and suppose $0 < p < 1$.

(*a*) if $\Sigma k^{p-1} L(k) S^2(k) a_k < \infty$, then $\Sigma a_k \sin kx$ converges, say to $g(x)$, and

$$x^{-p} L\left(\frac{1}{x}\right) S\left(\frac{1}{x}\right) g(x) \in L[0, \pi].$$

(*b*) if $\Sigma a_k \sin kx$ converges to $g(x)$ and

$$x^{-p} L\left(\frac{1}{x}\right) S\left(\frac{1}{x}\right) g(x) \in L(0, \pi),$$

then $\Sigma k^{p-1} L(k) S(k) a_k < \infty$.

Similar results hold for cosine series.

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